Subvarieties of Shimura varieties

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The aim of this lecture is to explain what the main theorem of the article [2] says, and to give an introduction to the formalism of Shimura varieties that is used in this and the next two lectures.

The main motivation for the specific form of the main theorem of [2] comes from an application to Wolfart's work [3] on algebraicity of values of hyper-geometric functions at algebraic arguments. So, in a way, one can say that this article was written especially for this conference.

The following conjecture is named after Yves André and Frans Oort; the terminology used in its statement and in the statement of the theorem following it will be explained in the first 45 minutes of the lecture.

Conjecture 1 (André–Oort)**.** *Let (G, X) be a Shimura datum, $K \subset G(\mathbb{A}_f)$ a compact open subgroup, and $Z \subset \mathrm{Sh}_K(G, X)_{\mathbb{C}}$ a closed algebraic subvariety that contains a Zariski dense set of special points. Then Z is special, i.e., of Hodge type.*

The main theorem of [2] is the following.

Theorem 2. *Let (G, X) be a Shimura datum, $K \subset G(\mathbb{A}_f)$ a compact open subgroup, and $Z \subset \mathrm{Sh}_K(G, X)_{\mathbb{C}}$ a closed algebraic curve that contains an infinite set Σ of special points. Let V be a finite dimensional faithful representation of G, and, for all $h \in X$, let V_h be the corresponding $\mathbb{Q}$-Hodge structure. For $x = \overline{(h, g)}$ in $\mathrm{Sh}_k(G, X)(\mathbb{C})$ let $[V_x]$ be the isomorphism class of V_h. Assume that all $[V_x]$, for x ranging through Σ, are equal. Then Z is special, i.e., of Hodge type.*

As explained in the previous lecture by Gisbert Wüstholz, Theorem 2 fills a gap in Wolfart's proof that if a certain hyper-geometric function has algebraic values at infinitely many algebraic arguments, then the monodromy group of that function is arithmetic. It is important to note that, in Wolfart's situation, the algebraic points at which the algebraic values are taken are known to correspond to abelian varieties with complex multiplication of a fixed type.

Wolfart's arguments involve certain explicit families of abelian varieties with certain types of endomorphisms, so one could think that the use of Shimura varieties in this lecture should be limited to the moduli spaces of precisely these kinds of abelian varieties. But there are at least two good reasons even in this case to use the general terminology of Shimura varieties. The first reason is that we get more flexibility: one can reduce the problem directly to the *smallest* Shimura variety containing Z, and one can reduce to the case $G = G^{\mathrm{ad}}$ (which, in terms of moduli interpretations, is more complicated than one would like). The second reason is that using the general formalism, as established by Deligne in [1], makes the situation actually a lot simpler (although more technical, maybe); the essential data are all encoded in the pair (G, X).

Let us now begin our explanation of what the statement of Theorem 2 above means. In this statement, G is a (connected) reductive linear algebraic group over $\mathbb{Q}$. We give two examples: $\mathrm{GL}_{2,\mathbb{Q}}$ and $\mathrm{GSp}_{2g,\mathbb{Q}}$ (symplectic similitudes).

The symbol X in the pair (G, X) is a $G(\mathbb{R})$-orbit in $\mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$, the set of morphisms of algebraic groups over $\mathbb{R}$. Here $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{\mathrm{m}}$. Hence, for every $\mathbb{R}$-algebra A, one has $\mathbb{S}(A) = (\mathbb{C} \otimes_{\mathbb{R}} A)^{\times} = \{\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right) \in \mathrm{GL}_2(A)\}$.

In the example $G = \mathrm{GL}_{2,\mathbb{Q}}$ one can take $X = \mathbb{H}^{\pm} = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})/\mathbb{S}(\mathbb{R})$.

The importance of $\mathbb{S}$ comes from its property that, for V a finite dimensional $\mathbb{R}$-vector space, to give a Hodge structure on V is the same as giving it an action by $\mathbb{S}$.

The pair (G, X) must satisfy the following three properties:

(1) $\forall h \in X$ the Hodge structure on $\mathrm{Lie}(G_{\mathbb{R}})$ is of type $\{(-1, 1), (0, 0), (1, -1)\}$;
(2) for all $h \in X$: $\mathrm{inn}_{h(i)}$ is a Cartan involution of $G^{\mathrm{ad}}_{\mathbb{R}}$, that is, the group $\{g \in G^{\mathrm{ad}}(\mathbb{C}) \mid h(i)\overline{g}h(i)^{-1} = g\}$ is compact;
(3) write $G^{\mathrm{ad}} = \prod_i G_i$ with G_i simple, then for all i and all $h \in X$, the induced morphism $\mathbb{S} \to G_{i,\mathbb{R}}$ is not trivial.

These conditions assure: X has a unique complex structure such that every representation V of $G_{\mathbb{R}}$ gives a variation of $\mathbb{R}$-HS on X; the connected components of X are hermitian symmetric domains (notation: X^+); $\pi_0(X)$ is finite, and $G(\mathbb{Q})$ acts transitively on it.

At this point, we know what (G, X) and V_h are.

The adèles. We let $\mathbb{A}_{\mathrm{f}} := \prod'_p \mathbb{Q}_p = \mathbb{Q} \otimes \hat{\mathbb{Z}}$, and $\mathbb{A} := \mathbb{A}_{\mathrm{f}} \times \mathbb{R}$; both are topological $\mathbb{Q}$-algebras, $\hat{\mathbb{Z}}$ is open in $\mathbb{A}_{\mathrm{f}}$ and has its own profinite topology. Then K in Theorem 2 is a compact open subgroup of $G(\mathbb{A}_{\mathrm{f}})$ (the topology on $G(\mathbb{A}_{\mathrm{f}})$ is obtained by embedding G as a *closed* subvariety of an affine N-space over $\mathbb{Q}$, and then restricting the topology of $\mathbb{A}_{\mathrm{f}}^N$). In the example $G = \mathrm{GL}_{2,\mathbb{Q}}$ one can take a maximal compact subgroup $\mathrm{GL}_2(\hat{\mathbb{Z}})$, or, more generally, for $n \in \mathbb{Z}$ non-zero, $\ker(\mathrm{GL}_2(\hat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}))$.

For (G, X) and K as above, one then defines:

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) := G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_{\mathrm{f}})/K).$$

The set $G(\mathbb{Q})\backslash(\pi_0(X) \times G(\mathbb{A}_{\mathrm{f}})/K)$ is finite; choosing representatives $([X^+], g_i)$ shows that $\mathrm{Sh}_K(G, X)(\mathbb{C}) = \coprod_i \Gamma_i\backslash X^+$, with the $\Gamma_i = G(\mathbb{Q})_{[X^+]} \cap g_i K g_i^{-1}$ arithmetic subgroups of $G(\mathbb{Q})$. Baily and Borel have shown that $\mathrm{Sh}_K(G, X)(\mathbb{C})$ is, (log) canonically, the complex analytic variety associated to a quasi-projective complex algebraic variety $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$. For example, $\mathrm{Sh}_{\mathrm{GSp}_{2g}(\hat{\mathbb{Z}})}(\mathrm{GSp}_{2g}, \mathbb{H}_g^{\pm})_{\mathbb{C}}$ is the moduli space $A_{g,1,\mathbb{C}}$ for complex principally polarised abelian varieties of dimension g.

Varying K gives a projective system of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$, with finite transition morphisms. One defines $\mathrm{Sh}(G, X)_{\mathbb{C}}$ as the projective limit of this system; it is a scheme, not locally of finite type, but say pro-algebraic. The reason to consider this limit is that $G(\mathbb{A}_{\mathrm{f}})$ acts on it, and one has, for $K \subset G(\mathbb{A}_{\mathrm{f}})$ open compact: $\mathrm{Sh}_K(G, X)_{\mathbb{C}} = (\mathrm{Sh}(G, X)_{\mathbb{C}})/K$. The $G(\mathbb{A}_{\mathrm{f}})$-action also gives a nice description of

Hecke correspondences. For K and K' open compact, and for $g \in G(\mathbb{A}_f)$, one has a correspondence $T_g\colon \mathrm{Sh}_K(G,X)_\mathbb{C} \leftarrow \mathrm{Sh}_{K\cap gK'g^{-1}}(G,X)_\mathbb{C} \to \mathrm{Sh}_{K'}(G,X)_\mathbb{C}$, induced by the action of g on $\mathrm{Sh}(G,X)_\mathbb{C}$.

Definition 3. *Let (G,X) be a Shimura datum, and $K \subset G(\mathbb{A}_f)$ a compact open subgroup. A closed subvariety S of $\mathrm{Sh}_K(G,X)_\mathbb{C}$ is called* special*, or* of Hodge type*, if there exists a Shimura datum (G',X'), a morphism $f\colon G' \to G$ such that for each $h \in X'$ one has $f \circ h \in X$, and an element $g \in G(\mathbb{A}_f)$, such that S is an irreducible component of the image of:*

$$\mathrm{Sh}(G',X')_\mathbb{C} \xrightarrow{f} \mathrm{Sh}(G,X)_\mathbb{C} \xrightarrow{\cdot g} \mathrm{Sh}(G,X)_\mathbb{C} \xrightarrow{q} \mathrm{Sh}_K(G,X)_\mathbb{C},$$

where q denotes the quotient morphism. A point s in $\mathrm{Sh}_K(G,X)_\mathbb{C}$ is called special *if it is a zero-dimensional special subvariety.*

For example, the special points in $A_{g,1,\mathbb{C}}$ are precisely the points that correspond to abelian varieties of CM-type.

At this point of the talk, the statement of Theorem 2 makes sense. However, the notion of Mumford-Tate group associated to a $\mathbb{Q}$-HS helps to understand it better. For (G,X) a Shimura datum and $h \in X$, one lets $\mathrm{MT}(h)$ be the smallest sub algebraic group H of G such that h factors through $H_\mathbb{R}$; these $\mathrm{MT}(h)$ are reductive. For P in $\mathrm{Sh}_K(G,X)_\mathbb{C}$ one then has an algebraic $\mathbb{Q}$-group $\mathrm{MT}(P)$ together with a $G(\mathbb{Q})$-conjugacy class of embeddings in G. A point P is then special if and only if $\mathrm{MT}(P)$ is commutative, i.e., a torus. For Z a closed subvariety of $\mathrm{Sh}_K(G,X)_\mathbb{C}$ there is then a *generic* Mumford-Tate group $\mathrm{MT}(Z)$ with the property that for all $P \in Z$ one has $\mathrm{MT}(P) \subset \mathrm{MT}(Z)$, with equality outside a countable union of proper closed subvarieties of Z; the points P where equality occurs are called *Hodge generic*. This $\mathrm{MT}(Z)$ gives us a description of the smallest special subvariety of $\mathrm{Sh}_K(G,X)_\mathbb{C}$ containing Z.

We can now give a very short description of the proof of Theorem 2. Let S be the connected component of $\mathrm{Sh}_K(G,X)_\mathbb{C}$ that contains Z. One first replaces the ambient Shimura variety by the smallest one containing Z, i.e., Z is then Hodge generic. The second step is to replace G by G^{ad}. Then comes the difficult step, where one invokes the help of Hecke and Galois. One manages to produce a Hecke correspondence T_q on S such that all $T_q + T_{q^{-1}}$-orbits in S are dense (for the Archimedean topology), and such that $T_q Z = Z = T_{q^{-1}} Z$. Then, of course, $Z = S$ and Z is special. In order to get $Z \subset T_q Z$ one uses the Galois orbits of elements in Σ: one can arrange for $Z \cap T_q Z$ containing such a Galois orbit that is larger than the degree allows if the intersection would be proper.

References

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[2] Bas Edixhoven, Andrei Yafaev, *Subvarieties of Shimura varieties.* Ann. of Math. (2) **157** (2003), no. 2, 621–645.

[3] Jürgen Wolfart, *Werte hypergeometrischer Funktionen.* Invent. Math. **92** (1988), no. 1, 187–216.