Summer School Alpbach
Motives, periods and transcendence

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Roughly, periods are complex numbers of the form

$$\int_Z \omega$$

where ω is a closed algebraic differential form on an algebraic variety X over $\mathbb{Q}$ (or maybe $\bar{\mathbb{Q}}$, maybe a differential form with log poles) and $Z \subset X(\mathbb{C})$ is a compact real submanifolds (maybe with boundary on an algebraic subvariety $D \subset X$). These numbers have very interesting properties from the point of view of number theory. In particular, there is a long history of investigation into their transcendence properties. For example,

$$\int_{S^1} \frac{dz}{z} = 2\pi i$$

is a period number, which was shown to be transcendental by Lindemann in 1882.

In this summer school we are going to concentrate on the formal properties of the set of all periods, following the ideas of Kontsevich [Ko]. The main reference will be [HMS] by Huber and Müller-Stach. Many basic details can be found in [Frh].

The notion of period is generalized to all values of the integration pairing between (relative) de Rham cohomology and (relative) singular homology. The period conjecture in the formulation of Kontsevich says that the only relations between these periods are the obvious one: induced by linearity, functoriality and boundary maps in long exact sequences. He makes this precise by introducing the notion of formal periods, subject to only these relations. The ring P of these formal periods (and hence conjecturally the set of periods) has a strong structural property:

Theorem:(Nori, [Ko]) Spec$P$ *is a torsor under the motivic Galois group.*

The aim of the summer school is to introduce all words in this Theorem and explain its proof as given in [HMS].

The first three section of the program are independent of each other. Section 4 relies heavily on section 3, to some extent on section 2, and a bit on section 1.

Speakers of any given section are encouraged to discuss among each other and shift material from one talk to the other if they feel this would be more suitable.

1 Classical periods

This section is meant as an unsystematic tour around examples of periods and transcendence results.

1.1 **periods for curves:** explicit examples, classical transcendence results

1.2 **Kontsevich-Zagier periods:** definition and examples, [KoZ], [Frh] 5.3, Section 8.

2 Cohomology

A series of talks leading to the definition of periods as values of the pairing between relative singular homology and de Rham cohomology

2.1 **singular cohomology**: definition of singular cohomology for pairs of topological spaces. The spaces that we need are the analytifications of algebraic varieties, in particular CW-complexes. Properties of singular cohomology: long exact sequence relative singular cohomology of a triple $X \supset Y \supset Z$ with proof; Künneth formula, Poincaré duality for singular cohomology of compact manifolds; singular cohomology can be computed by C^∞-chains; universal coefficient theorem (case of fields suffices). Give as many proofs as time allows. See [BT, W].

2.2 **singular cohomology as sheaf cohomology**: define sheaf cohomology on general topological spaces. Then concentrate on our case: analytification of algebraic varieties. State that singular cohomology with coefficients in an abelian group A can be computed as sheaf cohomology with coefficients in the constant sheaf A. Give a proof in the case of manifolds. Then extend the result to relative cohomology, with our without proof. The relationabstract is as follows. Let X be a complex analytic space, $D \subset X$ a closed subspace with open complement $j : U \to X$. Then

$$H^i(X, j_! A) = H^i(X, D; A)$$

where the left hand side is sheaf cohomology and $j_!$ is the extension by zero, i.e., the sheafification of the presheaf

$$V \mapsto \begin{cases} A & V \subset U \\ 0 & \text{else} \end{cases}$$

If you want to give a proof: the result follows easily by taking the long exact sequence for the short exact sequence of sheaves on X

$$0 \to j_! A \to A \to i_* A \to 0$$

($i : D \to X$ the closed embedding).

2.3 **Nori's basic lemma**: Formulate and prove Nori's basis lemma.

Theorem: Let X be an affine variety over $\mathbb{Q}$ of dimension d, $D \subsetneq X$ a closed subvariety. Then there is a closed subvariety $D' \subsetneq X$ containing D such that

$$H^i(X(\mathbb{C}), D'(\mathbb{C}); \mathbb{Q}) = 0 \qquad i \neq d$$

This talk is more difficult than the previous ones. It needs some algebraic geometry (blow-ups, divisors with normal crossings) and good grasp of singular cohomology. E.g. the proof needs a Poincare duality statement for relative singular cohomology of non-compact spaces which can best be understood by Verdier duality between $j_!$ and j_*. The reference is a sketch in talks by Nori [N], [N1]. There is an alternative proof by Beilinson in the language of perverse sheaves [B]. This does not seem suitable for our summer school.

2.4 **de Rham cohomology**: Define de Rham cohomology for complex manifolds as hypercohomology of the complex of holomorphic differential forms. Define de Rham cohomology for smooth algebraic varieties (over a field k of characteristic 0) as hypercohomology of the complex of algebraic differential forms. We want to understand that for a smooth projective variety X over $\mathbb{Q}$, there is a natural isomorphism:

$$H^i_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^i(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{C}$$

This is called the *period isomorphism*. The argument is a concatenation of properties of de Rham cohomology that we need to explain:

- base change formula for the change of ground field; this allows to pass from varieties over $\mathbb{Q}$ to varieties over $\mathbb{C}$
- GAGA, ie., the comparison between algebraic de Rham cohomology over $\mathbb{C}$ with holomorphic de Rham cohomology. Using the hypercohomology spectral sequence this reduced to the case of coherent sheaves and the comparison between algebraic and holomorphic differential forms.
- Formulate the holomorphic Poincare lemma and deduce that singular cohomology with coefficients in $\mathbb{C}$ agrees with holomorphic de Rham cohomology
- use the universal coefficient theorem to compare with singular cohomology with $\mathbb{Q}$-coefficients.

The details are explained quite nicely in [Frh] Section 2–4. Define periods of X as the $\mathbb{Q}$-span of the matrix entries of this isomorphism in rational bases on both sides (compare p-periods in [Frh]).

2.5 **relative de Rham cohomology**: We want to extend the last talk to the relative setting. For a pair X and a closed subvariety D, we need relative de Rham cohomology $H^i_{\mathrm{dR}}(X, D)$ such that

$$\dots \to H^i(X, D) \to H^i_{\mathrm{dR}}(X) \to H^i(D) \to H^{i+1}(X, D) \to \dots$$

is a long exact sequence compatible under the period isomorphism. Define periods of (X, D) as in the last talk.

We concentrate on a special case X smooth over $\mathbb{Q}$. First consider the case when D is also smooth. Then

$$H^i_{\mathrm{dR}}(X, D) = H^i\left(X, \mathrm{Cone}\left(\Omega_X \to i_*\Omega_D\right)[-1]\right)$$

has the correct property. More generally, let $D \subset X$ a strict divisor with normal crossings, ie., $D = \bigcup_{i=1} D_i$ and all D_i are smooth subvarieties of codimension 1 and moreover all intersections

$$\bigcap_{i \in I} D_i \qquad I \subset \{1, \dots, n\}$$

are smooth. Let $\tilde{D}$ be the Chech complex of the finite cover $\coprod D_i \to D$. Explicitly, let

$$\tilde{D}_m = \coprod_{1 \le i_0, \dots, i_m \in \le n} \bigcap_{j=0}^{m} D_{i_j}$$

In particular, $\tilde{D}_0 = \coprod_{i=1}^n D_i$ and then

$$\tilde{D}_m = \tilde{D}_0 \times_D \tilde{D}_0 \times_D \cdots \times_D \tilde{D}_0 \qquad m+1 \text{ factors}$$

There are natural boundary maps

$$\partial_i : \tilde{D}_m \to \tilde{D}_{m-1}$$

induced by projection away from the i-th coordinate.

Let $\pi : D_m \to X$ be the natural projection map. It is finite. Consider the double complex

$$\pi_*\Omega_{\tilde{D}_0} \to \pi_*\Omega_{\tilde{D}_1} \to \dots \to \pi_*\Omega_{\tilde{D}_n}$$

with boundary maps the alterning sum of the maps induced by the ∂_i. We put

$$\Omega_D = \mathrm{tot}\pi_*\Omega^*_{\tilde{D}_*}$$

the total complex of this double complex. Then

$$H^i_{\mathrm{dR}}(D) = H^i(X, \Omega_D) \qquad H^i_{\mathrm{dR}}(X, D) = H^i\left(X, \mathrm{Cone}\left(\Omega_X \to i_*\Omega_D\right)[-1]\right)$$

Note that from a combinatorial point of view this is the same structure as the sheafification of the Chech-complex of an open cover.

See [Frh] Section 3.

3 Tannaka Categories

This section is formal algebra or representation theory. The only algebraic geometry that enters is the appearance of affine algebraic groups.

3.1 **Tannaka duality**: Introduce the notion of tensor categories, duality, and fibre functors. Define neutral Tannakian categories. Define algebraic groups and Hopf algebras. We are only going to need the affine case. In this context the world algebraic means finite dimensional variety. We are also going to need pro-algebraic groups: those which are projective limits of algebraic ones. Equivalently, the Hopf algebras are direct limit of finite dimensional Hopf algebras. A good example is the multiplicative group $\mathbb{G}_m = \mathrm{Spec} k[T, T^{-1}]$.

    State the main theorem of Tannaka theory without proof: Every neutral Tannaka category is equivalent to the category of representations of a pro-algebraic group. This group is called the *Tannaka dual* of the category. A good running example is the category of graded vector spaces. It's Tannaka dual is $\mathbb{G}_m$. Be careful about the definition of the tensor product! Use the definition that makes the forgetful functor to vector spaces a fibre functor.

    The main reference for this talk is [DM].

3.2 **Nori's diagram category**: A diagram is an oriented graph. A representation assigns to every vertex a vector space and to every edge a linear map. Nori's theorem asserts that there is a universal abelian category attached to this situation. The category is equivalent to the category of comodules of a certain coalgebra. State the result precisely and give a sketch of proof. [vW] (*von Wangenheim*)

3.3 **multiplicative structures and localization**: Our next aim is to put a tensor structure on the diagram category of talk 3.2 or equivalently turn A into a bialgebra. Also explain briefly the notion of localization of a diagram and its effect on the bialgebra. [HMS] Appendix B.

3.4 **Nori's rigidity criterion**: We now want to establish duality on a tensor category or equivalently turn A into a Hopf algebra. Follow [HMS]. This talk needs a speaker comfortable with working with algebraic groups. [HMS] Appendix C.

3.5 **pairs of representations**: When given two rather than one representations of the diagram category, their comparison is measured in a formal period coalgebra. [HMS] 2.1-2.2, 2.4-2.6

4 Nori motives

This section brings together the results from the previous two: Define a particular diagram using pairs of algebraic varieties and a particular representation via relative singular cohomology. A survey of the theory can be found in [Le].

4.1 **diagrams of pairs**: Define the diagrams of pairs, good pairs and very good pairs: vertices are triples (X, D, i) with $i \in \mathbb{Z}$, X an algebraic variety over $\mathbb{Q}$ and D a closed subvariety; edges are induced by functoriality and boundary maps for triples. [HMS] Defn. 1.1 Singular cohomology and de Rham cohomology are representations (use talk 2.5). Define the category of Nori motives as its diagram category with respect to singular cohomology [HMS] Defn. 1.3. Explain why de Rham cohomology extends to all Nori motives using [HMS] Lemma 3.1. Define the formal period algebra [HMS] 2.4 (you do not know that it is an algebra). Define *period numbers* as the image of these under the evaluation map to $\mathbb{C}$. This talk uses 3.2 and 2.5. Mathematically, there is nothing complicated going on. The difficulty will be skipping around the various chapters and reading things out of order. If there is not enough material for a full talk, the next speaker could use some help.

4.2 **yoga of diagrams**: We want to define a tensor product on Nori motives. This is easily done on the subdiagram of good pairs. Explain using the basic lemma that the diagrams of pairs, good pairs and very good pairs define the same category. [HMS] Cor 1.6, Cor. 1.7, and App. D Formulate that the diagram of good pairs has a multiplicative representation [HMS] Thm 1.5 b)c). Also state Cor. 1.9.

    This talk uses talk 3.3. The speaker should have some experience with cohomology.

4.3 **rigidity**: Apply Nori's rigidity criterion to the category of Nori motives (talk 3.4) We now have established that Nori motives are a neutral Tannaka category. Define the motivic Galois group as its Tannaka dual. [HMS] 1.10-1.13.

    This talk uses talk 3.4 and some ideas from the proof of the basic lemma (talk 2.3). The speaker should have some experience with cohomology of algebraic varieties.

4.4 **torsors**: A torsor in our setting is a pro-algebraic group operating on some pro-algebraic scheme (projective limit of varieties of finite type) such that the operation is simply transitive on $\mathbb{C}$-values points. This is an example of a torsor for the flat topology.

    If time allows, discuss torsors abstractly [HMS] Appendix A and in the context of Tannaka duality [DM], [HMS] 3.2-3.3, 3.5. Present [HMS] Cor. 3.4. Apply the general theory to Nori motives with fibre functors de Rham cohomology, singular cohomology. Explain the proof of the main theorem and its consequences for period numbers. [HMS] 2.8., 2.10-2.12.

    Mathematically, this talk is simpler than the last two, but the speaker needs to have an overview about what was going on from beginng to end.

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