MULTIPLE ZETA VALUES

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The goal of this workshop, to be held in the beautiful Alpbach, 2–7 September 2012, is to study the motivic approach to multiple zeta values (MZV's) including the recent advances due to Francis Brown. Basically, we will follow the presentation of Deligne [7]. We also refer to the foundational paper [6] for many preliminary facts concerning the motivic fundamental group of $\mathbb{P}^1$ without three points and the appearance of MZV's in this context.

The program consists of 13 talks by 90 minutes. Each talk is provided with a detailed plan and description. All statements are expected to be proved or at least explained if the converse is not mentioned. For most of them, hints and ideas can be found in the text below as well as in the cited papers. The speakers of the workshop are urged to start preparing their talks as soon as possible. The preparation assumes a creative work with making sort of exercises and compiling various sources and melting them under common notation and concepts of the workshop. The division of talks is tentative, and the speakers of adjacent talks can exchange some of the material to be covered.

We assume familiarity with the following notions: fundamental groups, path spaces, filtered vector spaces, Betti cohomology, (algebraic) de Rham cohomology, mixed Hodge structures, periods, linear algebraic groups (mostly unipotent), Lie algebras, Hopf algebras, dg-algebras, total complexes of bicomplexes. Some familiarity with simplicial and cosimplicial objects, sheaves and local systems will be certainly helpful. The difficulty of most talks is rather conceptual than technical.

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Sunday. Introduction

1. Introduction to multiple zeta values

This talk contains an introduction to MZV's, a discussion of their general properties, and the statement of conjectures and results about MZV's that will be discussed during the workshop.

Definition: For a sequence of positive integers $\bar{n} = (n_1, \ldots, n_k)$, put

$$\zeta(\bar{n}) := \sum_{i_1 > \ldots > i_k > 0} \frac{1}{i_1^{n_1} \ldots i_k^{n_k}}.$$

This series converges iff $n_1 \geq 2$. A weight of $\zeta(\bar{n})$ is $|\bar{n}| := n_1 + \ldots + n_k$. The $\mathbb{Q}$-vector space $\mathcal{Z} \subset \mathbb{R}$ generated by MZV's is filtered by weight.

Stuffle relations: The $*$-product between sequences, stuffle relations. Corollary: $\mathcal{Z}$ is a filtered $\mathbb{Q}$-algebra. Example: $\zeta(2) \cdot \zeta(2) = 2\zeta(2,2) + \zeta(4)$.

Shuffle relations: Bijection between sequences $\bar{n}$ with $n_1 \geqslant 2$ and words $W(\bar{n})$ in 0 and 1 that start by 0 and end by 1. The group of shuffles, the shuffle product between sequences. Shuffle relations (without proof). Example: $\zeta(2) \cdot \zeta(2) = 2\zeta(2,2) + 4\zeta(3,1)$.

Regularized relations: Statement (without proof). Example: $\zeta(3) = \zeta(2,1)$.

Conjectures: All relations between MZV's are homogenous. Zagier's conjecture: dimensions d_n of the adjoint quotients of $\mathcal{Z}$ satisfy $d_n = d_{n-2} + d_{n-3}$. The ideal of relations between MZV's is generated by the above three sets of relations. Explicit examples: $d_0, \ldots, d_4$.

Statement of main results: Goncharov–Terasoma's theorem: $d_n \leqslant d_{n-2} + d_{n-3}$. Brown's theorem: all MZV's are linear combinations of $\zeta(\bar{n})$, where $n_i \in \{2, 3\}$.

References: [1, §§25.1, 25.2], [15, §§1, 8 ,10].

Monday. Chen's theorem

2. Iterated integrals

This talk presents MZV's as iterated integrals. As an application, the shuffle relations are obtained. Iterated integrals are interpreted as periods of certain relative cohomology and in terms of monodromy of a unipotent connection. At the end, one states Chen's theorem.

Definition: For $I := [0,1]$, consider the simplex $\sigma^n \subset I^n$ given by

$$\sigma^n := \{(t_1, \dots, t_n) | 1 \geq t_1 \geq \dots \geq t_n \geq 0\}\,.$$

For a smooth connected manifold M, a (piecewise) smooth path $\gamma\colon I \to M$, and a collection of complex valued smooth 1-forms $\bar{\omega} := (\omega_1 \dots \omega_n)$ on M, put

$$\int_\gamma \bar{\omega} := \int_{\sigma^n} (\gamma^n)^*(\omega_1 \boxtimes \dots \boxtimes \omega_n)\,,$$

where $\boxtimes$ denotes the exterior product of forms on M^n. Explicit description in terms of functions f_i on I such that $\gamma^*(\omega_i) = f_i dt$.

MZV's as iterated integrals: Put $M = \mathbb{P}^1(\mathbb{C}) \smallsetminus \{0, 1, \infty\}$, $\omega_0 := \frac{dz}{z}$, $\omega_1 := \frac{dz}{1-z}$. Define $\bar{\omega}(\bar{n})$ as a sequence of ω_0 and ω_1 encoded by the word $W(\bar{n})$ from Talk 1. For a path γ and $0 < \varepsilon \ll 1$, by γ_ε denote the segment of γ that starts at $\gamma(\varepsilon)$ and ends at $\gamma(1-\varepsilon)$. By dch denote the real interval $[0,1]$ considered as a path from 0 to 1 on $\mathbb{P}^1(\mathbb{C})$. Then there is an equality

$$\zeta(\bar{n}) = \lim_{\varepsilon \to 0} \int_{dch_\varepsilon} \bar{\omega}(\bar{n})\,.$$

Basic properties of iterated integrals: Product of integrals:

$$\int_\gamma \bar{\omega} \cdot \int_\gamma \bar{\omega}' = \sum_\tau \int_\gamma \tau(\bar{\omega}\bar{\omega}')\,,$$

where τ runs over shuffles. Composition of paths:

$$\int_{\gamma \circ \gamma'} \bar{\omega} = \sum_{i=0}^n \int_\gamma \omega_1 \dots \omega_i \cdot \int_{\gamma'} \omega_{i+1} \dots \omega_n.$$

The proof uses decompositions of (products of) the simplices σ^n. Corollary: shuffle relations between MZV's.

Differential of functions:

$$\int_\gamma (df)\omega_1 \dots \omega_n = \int_\gamma (f\omega_1)\omega_2 \dots \omega_n - f(a) \int_\gamma \omega_1 \dots \omega_n\,,$$

$$\int_\gamma \omega_1 \dots \omega_{i-1}(df)\omega_i \dots \omega_n = \int_\gamma \omega_1 \dots \omega_{i-1}(f\omega_i) \dots \omega_n - \int_\gamma \omega_1 \dots (f\omega_{i-1})\omega_i \dots \omega_n\,,$$

$$\int_\gamma \omega_1 \dots \omega_n(df) = f(b) \int_\gamma \omega_1 \dots \omega_n - \int_\gamma \omega_1 \dots \omega_{n-1}(f\omega_n)\,.$$

The proof uses Stokes theorem.

Taking the inverse of the path:

$$\int_\gamma \omega_1 \dots \omega_n = (-1)^n \int_{\gamma^{-1}} \omega_n \dots \omega_1 \,.$$

Corollary: duality relations between MZV's (in particular, $\zeta(3) = \zeta(2,1)$).

Cohomological interpretation: Given points a and b on M, let $Z^n_{a,b} \subset M^n$, $n \geqslant 1$, be the closed subset consisting of n-tuples $(x_1, \dots, x_n)$ such that one of the following equalities is satisfied: $a = x_1$, $x_1 = x_2$, $\dots$, $x_{n-1} = x_n$, or $x_n = b$. If $a = b$, then for short put $Z^n_a := Z^n_{a,b}$. To a path $\gamma\colon I \to M$ with $\gamma(0) = a$ and $\gamma(1) = b$, one associates a relative homology class in $H_n(M^n, Z^n_{a,b})$ by restricting the map $\gamma^n\colon I^n \to M^n$ to the simplex $\sigma^n \subset I^n$. This defines a linear map

$$\tilde{c}_n\colon \mathbb{Q}[\pi_1(M;a,b)] \to \mathbb{Q}_{a,b} \oplus H_n(M^n, Z^n_{a,b}), \quad n \geqslant 1\,,$$

where $\pi_1(M;a,b)$ denotes the set of homotopy classes of paths from a to b, $\mathbb{Q}_{a,b} := 0$ for $a \neq b$, and the map to $\mathbb{Q}_{a,a} := \mathbb{Q}$ is the augmentation.

Assume that a collection of closed 1-forms $\bar{\omega} = (\omega_1 \dots \omega_n)$ on M satisfies $\omega_i \wedge \omega_{i+1} = 0$ for $1 \leqslant i \leqslant n-1$. Then the exterior product of ω_i's defines a relative cohomology class in $H^n(M^n, Z^n_{a,b}; \mathbb{C})$ and there is an equality

$$\int_\gamma \bar{\omega} = \langle [\omega_1 \boxtimes \dots \boxtimes \omega_n], \tilde{c}_n(\gamma) \rangle \,.$$

Corollary: for $\bar{\omega}$ as above, the iterated integral $\int_\gamma \bar{\omega}$ is well-defined for the class of γ in $\pi_1(M;a,b)$.

Monodromy interpretation: Assuming that $\bar{\omega}$ is as in the previous section, consider a unipotent flat connection on the rank $n+1$ complex trivial bundle over M given by $d - N$, where a nilpotent matrix of 1-forms N is defined as follows:

$$N := \begin{pmatrix} 0 & \omega_1 & 0 & \dots & 0 \\ 0 & 0 & \omega_2 & \dots & 0 \\ & \vdots & & \dots & 0 \\ 0 & 0 & 0 & \dots & \omega_n \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Then the monodromy along γ of this connection is equal to the (finite) sum

$$\mathrm{Id} + \int_\gamma N + \int_\gamma N^{\otimes 2} + \dots + \int_\gamma N^{\otimes n} + \dots\,,$$

where $N^{\otimes n}$ denotes the power of the matrix N that involves the tensor product between forms (which has nothing to do with a tensor product of operators) and, by definition, $\int_\gamma \eta_1 \otimes \dots \otimes \eta_m := \int_\gamma \eta_1 \dots \eta_m$. This gives another proof of the composition of paths formula. Note that the analogous is true for any strict upper-triangular matrix N of 1-forms.

Chen's theorem: Given loops $\gamma_1 \dots, \gamma_{n+1}$ based at $a \in M$ and a path γ from a to b, one has the vanishing

$$\tilde{c}_n\big((1-\gamma_1) \circ \dots \circ (1-\gamma_{n+1}) \circ \gamma\big) = 0$$

in $H_n(M^n, Z^n_{a,b})$. The proof uses decomposition of the simplex σ_n into products of smaller simplices (similar to the proof of the composition of paths relation) and the inclusion-exclusion principle. Corollary: the map $\tilde{c}_n$ factors through a map

$$c_n\colon \mathbb{Q}[\pi_1(M;a,b)]/I^{n+1}\mathbb{Q}[\pi_1(M;a,b)] \to \mathbb{Q}_{a,b} \oplus H_n(M^n, Z^n_{a,b}), \quad n \geqslant 1\,,$$

where $I \subset \mathbb{Q}[\pi_1(M;a)]$ is the augmentation ideal. Chen's theorem states that c_n is an isomorphism (without proof).

References: [3, §§2.2, 3.3], [7, §1], [15, §7], [17, §6].

3. Pro-unipotent completion

The talk introduces the pro-unipotent completion (also known as Malcev completion) Γ^{un} of an abstract group Γ (assumed, for simplicity, to be finitely generated). This interprets Chen's theorem from Talk 2 as a calculation of the pro-unipotent completion of the fundamental group. The case of a free group is considered in more detail.

Unipotent groups and Hopf algebras: Definition of a pro-unipotent algebraic group scheme over a field (for short, a pro-unipotent group): every representation has an increasing unipotent filtration. Explicit description of the unipotent filtration on the Hopf algebra as the regular representation of the pro-unipotent group. Definition of a pro-unipotent group in terms of this filtration. Example: $\operatorname{Spec} T(V)$, where V is a vector space and $T(V) := \oplus_{n\geqslant 0} V^{\otimes n}$ with product given by shuffles and coproduct given by deconcatenation.

Equivalence between unipotent algebraic groups and finite-dimensional nilpotent Lie algebras over a field of characteristic zero: given $\mathfrak{g}$, put $\mathcal{O}(G)$ to be the topologically dual to the completion $U(\mathfrak{g})^{\wedge}$ of the universal enveloping algebra $U(\mathfrak{g})$ by the augmentation ideal. It follows that elements in G correspond to group-like elements in $U(\mathfrak{g})^{\wedge}$.

The abelianization $G/[G,G]$ of a unipotent algebraic group G is the additive group given by the k-vector space $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. The latter vector space is dual to $\mathrm{gr}^1_N\mathcal{O}(G)$, where N denotes the unipotent filtration (also, one has $\mathrm{gr}^1_N\mathcal{O}(G) \simeq H^1(G,k)$). The (iterated) coproduct map defines a canonical embedding of graded Hopf algebras

$$\mathrm{gr}^{\bullet}_N\mathcal{O}(G) \subset T\big(\mathrm{gr}^1_N\mathcal{O}(G)\big)\,.$$

For any G-torsor P the scheme $\operatorname{Spec}\big(\mathrm{gr}^{\bullet}_N\mathcal{O}(P)\big)$ is a (canonically) trivial torsor under $\operatorname{Spec}\big(\mathrm{gr}^{\bullet}_N\mathcal{O}(G)\big)$, whence one also has an embedding of graded algebras

$$\mathrm{gr}^{\bullet}_N\mathcal{O}(P) \subset T\big(\mathrm{gr}^1_N\mathcal{O}(G)\big)\,.$$

All the above has a version for pro-unipotent groups, which involves also a projective limit structure on $\mathfrak{g}$.

Pro-unipotent completion: Definition of Γ^{un} as a solution of the universal problem for group homomorphisms $\Gamma \to U(\mathbb{Q})$, where U is a pro-unipotent group over $\mathbb{Q}$ (in particular, Γ^{un} is a pro-unipotent group over $\mathbb{Q}$).

By Quillen, the Hopf algebra of regular functions on Γ^{un} is given by the formula

$$\mathcal{O}(\Gamma^{un}) = \varinjlim_n\, (\mathbb{Q}[\Gamma]/I^n)^{\vee}\,,$$

where $I \subset \mathbb{Q}[\Gamma]$ is the augmentation ideal and the product on the right hand side is dual to the map uniquely defined by $\gamma \mapsto 1 \otimes \gamma + \gamma \otimes 1$. In concrete way Γ^{un} may be given using the torsion free lower central series and explicit set of coordinates (this is probably the original construction of Malcev!). Correspondence between finite-dimensional representations of Γ^{un} and unipotent finite-dimensional representations of Γ, that is, representations of Γ that are extensions of trivial ones.

Given a torsor T under Γ, we obtain a torsor T^{un} under Γ^{un} with $\mathcal{O}(T^{un}) = \varinjlim (\mathbb{Q}[T]/I^n\mathbb{Q}[T])^\vee$. The obtained filtration on $\mathcal{O}(T^{un})$ equals the unipotent filtration induced by the action of Γ^{un}.

Interpretation of Chen's theorem: The isomorphisms c_n from Talk 2 induce a directed system structure on $H^n(M^n, Z^n_{a,b})$ and an isomorphism of vector spaces

$$c^\vee \colon \mathbb{Q}_{a,b} \oplus \varinjlim_n H^n(M^n, Z^n_{a,b}) \xrightarrow{\sim} \mathcal{O}\big(\pi_1(M; a, b)^{un}\big) .$$

The case of a free group: Let Γ be the free group on r generators $\gamma_1, \dots, \gamma_r$ (this is the fundamental group of $\mathbb{P}^1$ without several points and will be of fundamental importance later). The Lie algebra of Γ^{un} is the completion of the free graded Lie algebra over $\mathbb{Q}$ on generators $e_1, \dots, e_r$ in degree 1 by the lower central series. Its completed universal enveloping algebra is $\mathbb{Q}\langle\langle e_1, \dots, e_r\rangle\rangle$, the algebra of formal power series in non-commuting variables e_i, with the coproduct uniquely defined by $\Delta(e_i) = 1 \otimes e_i + e_i \otimes 1$. It follows that for any $\mathbb{Q}$-algebra R, the group $\Gamma^{un}(R)$ is the set of group-like elements in $R\langle\langle e_1, \dots, e_r\rangle\rangle$. This implies isomorphisms

$$\varprojlim_n \mathbb{Q}[\Gamma]/I^n \xrightarrow{\sim} \mathbb{Q}\langle\langle e_1, \dots, e_r\rangle\rangle, \quad \gamma_i \mapsto \exp(e_i) ,$$

$$\mathcal{O}(\Gamma^{un}) \xrightarrow{\sim} T(V) ,$$

where V is the dual to the $\mathbb{Q}$-vector space spanned by the variables e_i.

References: [6, App.], [7, §§2.1, 3.1], [12, §1]

4. Bar complex

Chen's isomorphism $c^\vee$ from Talk 3 induces a (Hopf) algebra structure on the direct limit $\mathbb{Q}_{a,b} \oplus \varinjlim H^n(M^n, Z^n_{a,b})$. The goal of this talk is to show how this structure can be defined independently in terms of cohomology, which will lead later to additional structures on the pro-unipotent completion of the fundamental group. The construction is based on the bar complex. Its conceptual explanation involves the cosimplicial path space, which is a combinatorial approximation to the path space with the topological interval I being replaced by the simplicial interval $\Delta[1]$.

Definition: Given a dg-algebra $A^\bullet$ over a field k and two morphisms of dg-algebras $a, b \colon A^\bullet \to k$, one defines the bar complex $B(A^\bullet; a, b)$ using the $\oplus$-total complex of the bicomplex

$$\ldots \to (A^\bullet)^{\otimes n} \to \ldots \to (A^\bullet)^{\otimes 2} \to A^\bullet \to k \to 0 ,$$

where the differential involves a, b, and multiplication in $A^\bullet$. If $a=b$, then for short put $B(A^\bullet;a):=B(A^\bullet;a,a)$. Deconcatenation defines a coproduct map

$$B(A^\bullet;a,c)\to B(A^\bullet;a,b)\otimes_k B(A^\bullet;b,c)\,,$$

which is a morphism of complexes. One also has a counit and a coinverse. A commutative product on $B(A^\bullet;a,b)$ is given by shuffles for commutative $A^\bullet$ (there are signs involved!). Example: if $A^\bullet=k\oplus V[-1]$ and a is the natural morphism, then $B(A^\bullet;a)$ is $T(V)$ from Talk 3.

Reduced bar complex: Assume that $A^{<0}=0$ and $H^0(A^\bullet)=k$. Define the reduced bar complex $\overline{B}(A^\bullet;a,b)$ as the quotient of $B(A^\bullet;a,b)$ over the subcomplex spanned by decomposable tensors such that one of the multiples has degree zero and their (total) differentials. The reduced bar complex has also product and coproduct structures. Explicit description of the reduced bar complex (especially of its H^0). The quotient map $B(A^\bullet;a,b)\to\overline{B}(A^\bullet;a,b)$ is a quasiisomorphism. To prove this one first uses a simplicial structure on the bar complex and applies normalization taking quotient over constants. Then one uses the quasiisomorphism $A^\bullet/k\to A^\bullet/(A^0\to dA^0)$.

Relation with iterated integrals: Let M be a smooth connected manifold, $A^\bullet_M$ be the complex valued de Rham complex (what follows is true for the real valued de Rham complex as well). Two points $a,b\in M$ define morphisms of dg-algebras from $A^\bullet_M$ to $\mathbb{C}$, denoted similarly. Basic properties of iterated integrals from Talk 2 give a map

$$iter\colon \mathbb{Q}[\pi_1(M;a,b)]\to H^0\big(\overline{B}(A^\bullet_M;a,b)\big)^\vee=H^0\big(B(A^\bullet_M;a,b)\big)^\vee,$$

$$\gamma\mapsto\Big\{\omega_1\otimes\ldots\otimes\omega_n\mapsto\textstyle\int_\gamma\omega_1\ldots\omega_n\Big\}$$

and, dually, a morphism of algebras that respects coproducts

$$iter^\vee\colon H^0\big(B(A^\bullet_M;a,b)\big)\to\mathbb{C}\otimes_{\mathbb{Q}}\mathcal{O}\big(\pi_1(M;a,b)^{un}\big)\,.$$

Comparison with relative cohomology: Let $K^\bullet_n$ be the total complex of the following truncation of the bar complex:

$$0\to(A^\bullet_M)^{\otimes n}\to(A^\bullet_M)^{\otimes(n-1)}\to\ldots\to(A^\bullet_M)^{\otimes m}\to\ldots\to k\to 0\,.$$

In particular, $B(A^\bullet_M)=\varinjlim K^\bullet_n$.

By Künneth and the simplicial resolution of the boundary $Z^n_{a,b}$, the relative cohomology $\mathbb{C}_{a,b}\oplus H^n(M^n,Z^n_{a,b};\mathbb{C})$ from Talk 2 are computed by the total complex $\widetilde{K}^\bullet_n$ of the bicomplex

$$0\to(A^\bullet_M)^{\otimes n}\to\bigoplus_{i=0}^{n}(A^\bullet_M)^{\otimes(n-1)}\to\ldots\to\bigoplus_I(A^\bullet_M)^{\otimes m}\to\ldots\to k^{\oplus(n+1)}\to 0\,,$$

where the m'th tensor powers are summed over all subsets $I\subset\{0,\ldots,n\}$ of $n-m$ elements.

Taking the sum defines a quasiisomorphism of complexes $\widetilde{K}^\bullet_n\to K^\bullet_n$ (this is a particular case of a general statement about simplicial objects in idempotent complete additive categories, whose proof is not required). This implies that Chen's theorem is equivalent to the fact that $iter^\vee$ is an isomorphism.

Cosimplicial path space: This section is optional. Define the cosimplicial manifold

$$P^\bullet(M):=\mathrm{Hom}(\Delta[1],M)\,.$$

Explicitly, one has $P^n(M) = M^{n+2}$, $n \geqslant 0$, with coboundary maps expressed in terms of the diagonal map $M \to M \times M$. There is a natural morphism from $P^\bullet(M)$ to the constant cosimplicial manifold $M \times M$. Given two points $a, b \in M$, one defines the cosimplicial topological space $P^\bullet(M; a, b)$ to be its fiber at (a, b). By Künneth, there is a canonical isomorphism

$$H^i\big(P^\bullet(M; a, b), \mathbb{C}\big) \simeq H^i\big(B(A^\bullet_M; a, b)\big)\,,$$

where cohomology of a cosimplicial manifold are defined using the $\oplus$-total complex of the corresponding bicomplex. The isomorphism respects the product and coproduct structures.

By $|E^\bullet| := \mathrm{Hom}(\sigma^\bullet, E^\bullet)$ denote the geometric realization of a cosimplicial topological space $E^\bullet$, where $\sigma^\bullet$ is the standard cosimplicial space formed by simplices σ^n and Hom is taken in the category of cosimplicial topological spaces. One has that $|P^\bullet(M; a, b)|$ is homeomorphic to the fiber of $\mathrm{Hom}(|\Delta[1]|, M)$ over $(a, b) \in M \times M$. The latter is homotopic to the usual path space $P(M; a, b)$, whose H^0 is $\mathbb{Q}[\pi_1(M; a, b)]^\vee$. With this, Chen's theorem states that the natural map

$$H^0\big(P^\bullet(M; a, b)\big) \to H^0\big(|P^\bullet(M; a, b)|\big)$$

is injective and its image is identified with the subspace of linear functionals on $H_0\big(|P^\bullet(M; a, b)|\big) = \mathbb{Q}[\pi_1(M; a, b)]$ that vanish on $I^n\mathbb{Q}[\pi_1(M; a, b)]$ for a sufficiently large n.

References: [6, §§3.7, 3.9, 3.10, 3.11], [9], [13, §8.3.3], [14, §§4, 5]

Tuesday. Hodge structure on the fundamental group

5. Proof of Chen's theorem

The talk consists in the proof of Chen's theorem in the equivalent formulation from Talk 4. One uses essentially the Riemann–Hilbert correspondence and the monodromy interpretation of iterated integrals. An alternative proof of Chen's theorem, due to Beilinson, is reproduced in [6, §§3.4, 3.8].

Isomorphisms between Hopf algebras: We need to prove that the morphism of algebras $iter^\vee$ from Talk 4 is an isomorphism. Since both algebras correspond to torsors under Hopf algebras, it is enough to prove that the morphism of Hopf algebras

$$iter^\vee\colon H^0\big(B(A^\bullet_M;a)\big)\to \mathbb{C}\otimes_{\mathbb{Q}}\mathcal{O}\big(\pi_1(M;a)^{un}\big)$$

is an isomorphism. For short, put $H(M;a):=H^0\big(B(A^\bullet_M;a)\big)$. By the definition of a pro-unipotent completion, $iter^\vee$ induces a functor

$$\Phi\colon \mathrm{Comod}\big(H(M;a)\big)\to \mathrm{Rep}^{un}\big(\pi_1(M;a)\big)\,.$$

By a general statement about Hopf algebras (which is a part of the Tannakian formalism from Talk 8), it is enough to show that Φ is an equivalence of categories.

Riemann–Hilbert correspondence: By $\mathrm{Conn}(M)$ denote the category of complex vector bundles on M with a flat connection. One has the functor from $\mathrm{Conn}(M)$ to $\mathrm{Rep}\big(\pi_1(M;a)\big)$ that sends (E,∇) to the fiber E_a at a with the monodromy representation. This is an equivalence of categories with a quasiinverse functor RH being constructed as follows. Given a finite-dimensional complex representation of $\pi_1(M;a)$, let L be the corresponding local system on M. The sheaf of sections $\mathcal{E}$ of E is then $L\otimes_{\mathbb{C}}\mathcal{A}^0_M$, where $\mathcal{A}^0_M$ denotes the sheaf of complex valued smooth functions on M. The connection on E is induced by the de Rham differential $d:\mathcal{A}^0_M\to\mathcal{A}^1_M$.

Unipotent connections: By $\mathrm{Conn}^{un}(M)$ denote the category of vector bundles with a unipotent flat connection. Note that we have an equivalence of categories

$$RH:\mathrm{Rep}^{un}\big(\pi_1(M;a)\big)\to \mathrm{Conn}^{un}(M)\,.$$

Let (E,∇) be in $\mathrm{Conn}^{un}(M)$. Then E is a trivial bundle being an extension of trivial bundles. Choose an isomorphism $\mathcal{E}\simeq E_a\otimes_{\mathbb{C}}\mathcal{A}^0_M$ that induces an identity on the (closed) fiber E_a at a. Then (E,∇) is uniquely defined by an element $N\in\mathrm{End}_{\mathbb{C}}(E_a)\otimes_{\mathbb{C}}A^1_M$, where $\nabla=d-N$. Since ∇ is unipotent, N is strict upper-triangular in a suitable basis in E_a. Therefore, $N^{\otimes n}=0$ for a sufficiently large n, where, as in Talk 2, $N^{\otimes n}$ involves the usual product between operators in $\mathrm{End}_{\mathbb{C}}(E_a)$ and the tensor products between forms.

From connections to comodules: Our aim is to construct a quasiinverse functor to Φ. The monodromy interpretation of iterated integrals predicts the following construction. For N as above, consider the (finite) sum

$$P:=\mathrm{Id}+N+N^{\otimes 2}+\ldots+N^{\otimes n}+\ldots\in\mathrm{End}_{\mathbb{C}}(E_a)\otimes_{\mathbb{C}}B(A^\bullet_M;a)^0\,.$$

The flatness condition on ∇ (that is, the equality $dN=N\wedge N$) implies that P is a cocycle in the bar complex. Thus, we obtain the class

$$[P]\in\mathrm{End}_{\mathbb{C}}(E_a)\otimes_{\mathbb{C}}H(M;a)\,.$$

One checks directly that $[P]$ defines a comodule structure on E_a over $H(M;a)$.

Functoriality of the construction: Let $f : (E,\nabla_E) \to (F,\nabla_F)$ be a morphism in $\mathrm{Conn}^{un}(M)$. By $f_a : E_a \to F_a$ denote the fiber of f at a. Since f commutes with ∇ (that is, $N_F \cdot f - f \cdot N_E = df$), the following equality holds in $\mathrm{End}_{\mathbb{C}}(E_a) \otimes_{\mathbb{C}} B(A^\bullet_M;a)^0$:

$$P_F \cdot f_a - f_a \cdot P_E = d\left(\sum_{n\geqslant 0}\sum_{i=0}^{n} N_F^{\otimes i} \otimes f \otimes N_E^{\otimes(n-i)}\right),$$

where d denotes the differential in the bar complex. This implies that the class $[P]$ does not depend on the choice of a trivialization of E as above and one has the functor

$$\mathrm{Conn}^{un}(M) \to \mathrm{Comod}\big(H(M;a)\big), \quad (E,\nabla) \mapsto (E_a,[P])\,.$$

Consider the composition of RH with this functor:

$$\Psi\colon \mathrm{Rep}^{un}\big(\pi_1(M;a)\big) \to \mathrm{Comod}\big(H(M;a)\big)\,.$$

It follows from the monodromy interpretation of iterated integrals that $\Phi\circ\Psi$ is isomorphic to the identity.

Essential surjectivity of Ψ: Choose a decomposition $A^1_M = d(A^0_M) \oplus V$. Then the dg-algebra $A^\bullet$

$$0 \to \mathbb{C} \xrightarrow{0} V \xrightarrow{d} A^2_M \xrightarrow{d} A^3_M \xrightarrow{d} \ldots$$

is a quasiisomorphic dg-subalgebra in $A^\bullet_M$. This implies that the corresponding bar complexes are quasiisomorphic (it is important here that we use the $\oplus$-total complex), whence $H^0\big(B(A^\bullet;a)\big) \simeq H(M;a)$.

On the other hand, by the explicit description of the reduced bar complex, $H^0\big(B(A^\bullet;a)\big)$ is a Hopf subalgebra in $T(V)$. It follows from Talk 3 that any comodule over $T(V)$ is given by a matrix of the form $\mathrm{Id} + N + N^{\otimes 2} + \ldots$ where N is a strict upper-triangular matrix with values in V. This implies that Ψ is essentially surjective.

Finally, by an abstract nonsense, the facts that $\Phi\circ\Psi \simeq \mathrm{Id}$, Ψ is essentially surjective, and Φ is faithful imply that $\Psi\circ\Phi \simeq \mathrm{Id}$.

6. Mixed Hodge structure on the fundamental group

The aim of this talk is to show that the (Hopf) algebra structure on the relative cohomology group $\mathbb{Q}_{a,b} \oplus \varinjlim H^n(M^n, Z^n_{a,b})$ respects the mixed Hodge structure on it when M is the set of complex points of an algebraic variety. The main idea is that the bar complex from Talk 4 has a geometric origin. This allows to define a mixed Hodge structure on the ring of regular functions on the pro-unipotent completion of the fundamental group.

Algebraic geometry over a tensor category: Let $\mathcal{C}$ be a symmetric tensor category. By $\mathrm{Ind}(\mathcal{C})$ denote the category of ind-objects in $\mathcal{C}$, that is, directed systems of objects in $\mathcal{C}$. Following Deligne, one calls a $\mathcal{C}$-affine scheme T an object in the opposite category of the category of commutative algebras in $\mathrm{Ind}(\mathcal{C})$, that is, $\mathcal{O}(T)$ is a commutative algebra ind-object in $\mathcal{C}$. Similarly, one can speak about $\mathcal{C}$-affine algebraic groups (or groupoids), etc. Given a symmetric tensor functor $\omega\colon \mathcal{C} \to \mathrm{Vect}(k)$ over a field k, one obtains actual affine schemes, affine algebraic groups, etc over k.

Mixed Hodge structures: In what follows one fixes an embedding of fields $k \subset \mathbb{C}$. By a mixed Hodge structure over k we mean a rational mixed Hodge structure such that its de Rham realization together with the Hodge and weight filtrations are defined over k. Periods of mixed Hodge structures. Examples: cohomology $H^n(X)$ (or homology) of an algebraic variety X over k, relative cohomology $H^n(X,Y)$ (or homology) for a closed subvariety $Y \subset X$ over k, $\mathbb{Q}(0)_H := H_0(\mathrm{Spec}(k))$, $\mathbb{Q}(1)_H := H_2(\mathbb{P}^1)$ (its weight is -2 and its period is $(2\pi i)^{-1}$). The category $\mathrm{MH}(k)$ of mixed Hodge structures over k is a symmetric tensor abelian category. One has de Rham and Betti realization functors

$$\omega_{dR}\colon \mathrm{MH}(k) \to \mathrm{Vect}(k), \quad \omega_B\colon \mathrm{MH}(k) \to \mathrm{Vect}(\mathbb{Q})$$

and a canonical comparison isomorphism $comp\colon \mathbb{C} \otimes_k \omega_{dR} \xrightarrow{\sim} \mathbb{C} \otimes_{\mathbb{Q}} \omega_B$. A mixed Hodge structure is of mixed Tate type if its weight adjoint quotients are direct sums of $\mathbb{Q}(n)_H$, $n \in \mathbb{Z}$ (by definition, $\mathbb{Q}(-1)_H := \mathbb{Q}(1)_H^\vee$). By $\mathrm{MTH}(k)$ denote the full subcategory in $\mathrm{MH}(k)$ that consists of mixed Tate Hodge structure over k.

Geometric origin of the Hopf structure: The crucial remark is that the differential, the product, and the coproduct structures on the bar complex $B(A^\bullet_M; a, b)$ are linear combinations of pull-backs along the diagonal map $M \to M \times M$ and the embeddings $a, b\colon \mathrm{pt} \to M$ (see Talk 4). Similarly, the isomorphism between relative cohomology $H^n(M^n, Z^n_{a,b})$ and cohomology of the truncated bar complex is of geometric origin, whence the directed system structure on $H^n(M^n, Z^n_{a,b})$ is of geometric origin as well.

Mixed Hodge structure on the fundamental group: Let X be a smooth algebraic variety defined over $k \subset \mathbb{C}$ such that $X(\mathbb{C})$ is connected, let $a, b \in X(k)$, and let $M = X(\mathbb{C})$. Then $Z^n_{a,b}$ is the set of complex points of an algebraic subvariety $Y^n_{a,b}$ in X^n over k. It follows from the above that $\mathbb{Q}_{a,b} \oplus H^n(X^n, Y^n_{a,b})$, $n \geqslant 1$, form a directed system in the category $\mathrm{MH}(k)$. This defines mixed Hodge affine schemes $\pi_1(X; a, b)_H$ together with a groupoid structure in $\mathrm{MH}(k)$

$$\pi_1(X; a, b)_H \times \pi_1(X; b, c)_H \to \pi_1(X; a, c)_H.$$

(An alternative short explanation is that $\mathcal{O}\big(\pi_1(X; a, b)_H\big)$ is equal to H^0 of the cosimplicial path variety $P^\bullet(X; a, b)$.)

De Rham and Betti and realizations: Applying ω_{dR} to $\pi_1(X; a, b)_H$, one obtains an affine scheme $\pi_1(X; a, b)_{dR} \simeq \mathrm{Spec}\, H^0\big(B(A^\bullet_X; a, b)\big)$ and a pro-unipotent group $\pi_1(X; a)_{dR}$ over k, where $A^\bullet_X$ is a suitable $R\Gamma$ of the algebraic de Rham complex of X.

Applying ω_B, one obtains an affine scheme $\pi_1(X; a, b)_B \simeq \mathrm{Spec}\, H^0\big(B(C^\bullet_{X(\mathbb{C})}; a, b)\big)$ and a pro-unipotent group $\pi_1(X; a)_B$ over $\mathbb{Q}$, where $C^\bullet_{X(\mathbb{C})}$ is the singular cochain complex of $X(\mathbb{C})$ (notice that it has a commutative dg-algebra structure). By Chen's theorem,

$$\pi_1(X; a)_B \simeq \pi_1(X(\mathbb{C}); a)^{un}, \quad \pi_1(X; a, b)_B \simeq \pi_1(X(\mathbb{C}); a, b)^{un}\,.$$

In particular, there is a canonical map $\pi_1(X(\mathbb{C}); a, b) \to \pi_1(X; a, b)_B(\mathbb{Q})$.

Comparison isomorphism: It follows that there is a canonical comparison isomorphism

$$comp\colon \mathbb{C} \otimes_k \mathcal{O}\left(\pi_1(X; a, b)_{dR}\right) \xrightarrow{\sim} \mathbb{C} \otimes_{\mathbb{Q}} \mathcal{O}\left(\pi_1(X; a, b)_B\right)\,.$$

Assume that a collection of closed algebraic 1-forms $\omega_1, \ldots, \omega_n$ on X over k satisfies $\omega_i \wedge \omega_{i+1} = 0$ for $1 \leqslant i \leqslant n-1$. Then $\omega_1 \otimes \ldots \otimes \omega_n$ defines an algebraic function on $\pi_1(X;a,b)_{dR}$ over k. Applying *comp*, we obtain a complex valued algebraic function on $\pi_1(X;a,b)_B$. By construction, *comp* is induced by $iter^\vee$ from Talk 4, that is, we have

$$comp(\omega_1\otimes\ldots\otimes\omega_n)(\gamma)=\int_\gamma \omega_1\ldots\omega_n$$

for any path γ from a to b.

Unipotent filtration: One has a well-defined unipotent filtration N on $\mathcal{O}\big(\pi_1(X;a,b)_H\big)$ in the category $\mathrm{MH}(k)$ induced by the action of $\pi_1(X;a)_H$. It follows that $\mathrm{gr}^1_N\mathcal{O}\big(\pi_1(X;a,b)_H\big)\cong H^1(X)$, whence, by Talk 3, one has an embedding of graded (Hopf) algebras $\mathrm{gr}^\bullet_N\mathcal{O}\big(\pi_1(X;a,b)_H\big)\subset T\big(H^1(X)\big)$ (both are ind-objects in $\mathrm{MH}(k)$). In particular, $\mathcal{O}\big(\pi_1(X;a,b)_H\big)$ has non-negative weights.

Example: $H^1(X)$ is pure of weight 2 iff $H^1(X)\cong\mathbb{Q}(-1)^{\oplus r}$ iff $H^1(\bar{X},\mathcal{O}_{\bar{X}})=0$, where $\bar{X}$ is a (any) smooth compactification of X. In this case the weight filtration on $\mathcal{O}\big(\pi_1(X;a,b)_H\big)$ coincides with the unipotent filtration and $\mathcal{O}\big(\pi_1(X;a,b)_H\big)$ is of mixed Tate type.

References: [4, §12.1], [5, §7.8], [7, §§2, 5.3]

7. The case of $\mathbb{P}^1$ without several points

In this talk one applies the set-up of Talk 6 to the case of $\mathbb{P}^1$ without several points. One shows that the corresponding mixed Hodge structure is of mixed Tate type. One also considers tangential base points and relates the corresponding periods with MZV's.

Mixed Hodge structure on the fundamental group: Let $D\subset\mathbb{P}^1$ be a reduced divisor defined over $k\subset\mathbb{C}$, $|D(\mathbb{C})|=r+1$, $X=\mathbb{P}^1\smallsetminus D$, and let $a,b\in X(k)$. By Ω denote the k-vector space of algebraic 1-forms on X with at most first order poles along D. In particular, $\dim_k(\Omega)=r$ and $\Omega\simeq H^1_{dR}(X)$. The de Rham complex $A^\bullet_X$ is quasiisomorphic to its subcomplex $k\oplus\Omega[-1]$. This shows that $\pi_1(X;a,b)_{dR}$ does not depend on points a and b and one has an isomorphism of (Hopf) algebras $\mathcal{O}\big(\pi_1(X;a,b)_{dR}\big)\simeq T(\Omega)$. In particular, the (Hopf) algebra $\mathcal{O}(\pi_1(X;a,b)_{dR})$ is naturally graded. The weight filtration is given by

$$W_{2n}\mathcal{O}\big(\pi_1(X;a,b)_{dR}\big)=\bigoplus_{i\leqslant n}\Omega^{\otimes i}$$

and the Hodge filtration is given by

$$F^p\mathcal{O}\big(\pi_1(X;a,b)_{dR}\big)=\bigoplus_{i\geqslant p}\Omega^{\otimes i}\,.$$

By Talks 3 and 6, one has an isomorphism of (Hopf) algebras $\mathcal{O}\big(\pi_1(X;a,b)_B\big)\simeq T(V)$, where V is a $\mathbb{Q}$-vector space of dimension r. The comparison isomorphism

$$comp\colon \mathbb{C}\otimes_k T(\Omega)\xrightarrow{\sim}\mathbb{C}\otimes_{\mathbb{Q}}T(V)$$

is encoded by iterated integrals as explained in Talk 6. Moreover, $\mathcal{O}\big(\pi_1(X;a,b)_H\big)$ is of mixed Tate type. To show this one either analyzes (relative) cohomology of X^n, or uses results from the end of Talk 6.

The case $D = \{0,\infty\}$: One has $\Omega = \frac{dz}{z}\cdot k$, $\pi_1(X;a)_B$ does not depend on a being commutative, and $\pi_1(X;a)_H \cong T\big(\mathbb{Q}(-1)_H\big)$. To deduce the last isomorphism one computes explicitly the comparison isomorphism *comp* using the description of the map $\Gamma \to \Gamma^{un}(\mathbb{Q})$, $\Gamma = \mathbb{Z}$, from Talk 3 and the equation

$$\int_\gamma \omega^{\otimes n} = \frac{1}{n!}\left(\int_\gamma \omega\right)^n .$$

In particular, $\pi_1(X;a)_{dR}$ is the additive group $\mathbb{G}_a$ and the comparison isomorphism identifies $\pi_1(X;a)_B(\mathbb{Q})$ with $2\pi i\mathbb{Q} \subset \mathbb{G}_a(\mathbb{C})$. By abuse of notation, one denotes the $\mathrm{MH}(k)$-algebraic group that corresponds to $T\big(\mathbb{Q}(-1)_H\big)$ also by $\mathbb{Q}(1)_H$.

Tangential base points: We continue with $X = \mathbb{P}^1 \smallsetminus D$ (actually, what follows has a version with $\mathbb{P}^1$ being replaced by any smooth projective curve over k). Given points $x, y \in D(k)$ and non-zero tangent vectors $u \in T_x\mathbb{P}^1$, $v \in T_y\mathbb{P}^1$, let $\pi_1(X(\mathbb{C}); u, v)$ denote the set of homotopy classes of (piecewise) smooth paths from x to y with the tangent vector at x being u and at y being $-v$ (we fix a coordinate on $I = [0,1]$). One has a canonical small counterclockwise loop $\gamma_x \in \pi_1(X(\mathbb{C}); u)$ "almost around" x.

Given a complex vector bundle E on $\mathbb{P}^1$ and a unipotent connection ∇ on E with at most first order poles along D, one defines a monodromy along $\gamma \in \pi_1(X(\mathbb{C}); u, v)$ by the regularization

$$\int_\gamma \nabla := \lim_{\varepsilon\to 0} \varepsilon^{\mathrm{res}_y(\nabla)} \circ \int_{\gamma_\varepsilon} \nabla \circ \varepsilon^{-\mathrm{res}_x(\nabla)} ,$$

where γ_ε is defined as in Talk 2, one has $\mathrm{res}(d-N) := \mathrm{res}(-N)$, and $\varepsilon^U := \exp(\log(\varepsilon)\cdot U)$ for a nilpotent matrix U (a local calculation shows that the limit exists). Example: $\int_{\gamma_x} \nabla = \exp(2\pi i \cdot \mathrm{res}_x(\nabla))$.

Mixed Hodge structure on the tangential fundamental group: Using the monodromy interpretation of iterated integrals from Talk 2, one defines $\int_\gamma \omega_1 \ldots \omega_n$ and a map

$$\begin{gathered} iter\colon \mathbb{Q}[\pi_1(X(\mathbb{C}); u, v)] \to \mathbb{C} \otimes_k T(\Omega)^\vee , \\ \gamma \mapsto \left\{\omega_1 \otimes \ldots \otimes \omega_n \mapsto \int_\gamma \omega_1 \ldots \omega_n\right\} . \end{gathered}$$

Explicitly, one has

$$\int_\gamma \omega_1 \ldots \omega_n = \lim_{\varepsilon\to 0} \sum_{0\leqslant i\leqslant j\leqslant n} \frac{(-1)^i}{i!(n-j)!} \prod_{l=1}^{i} \mathrm{res}_y(\omega_l)\cdot \int_{\gamma_\varepsilon} \omega_{i+1}\ldots\omega_{i+j} \cdot \prod_{l=j+1}^{n} \mathrm{res}_x(\omega_l)\cdot \log(\varepsilon)^{i+n-j} .$$

The monodromy interpretation implies the composition of paths formula and the above explicit description implies the product of integrals formula.

All the above has a version when one of the base points is an actual point in $X(k)$. It follows that for $a \in X(k)$, *iter* induces a morphism of torsors

$$\mathbb{C} \times_{\mathbb{Q}} \pi_1(X(\mathbb{C}); a, v)^{un} \to \mathbb{C} \times_k \operatorname{Spec} T(\Omega) ,$$

which respects the isomorphism of pro-unipotent groups

$$\mathbb{C} \times_{\mathbb{Q}} \pi_1(X(\mathbb{C}); a)^{un} \simeq \mathbb{C} \times_{\mathbb{Q}} \pi_1(X; a)_B \to \mathbb{C} \times_k \pi_1(X; a)_{dR} \simeq \mathbb{C} \times_k \operatorname{Spec} T(\Omega) .$$

Therefore, this is an isomorphism and letting the weight filtration on $\mathcal{O}\big(\pi_1(X(\mathbb{C});a,v)^{un}\big)$ to be the unipotent filtration, one obtains a torsor $\pi_1(X;a,v)_H$ in $\mathrm{MTH}(k)$ under $\pi_1(X;a)_H$. Further, taking products of torsors leads to mixed Tate Hodge affine schemes $\pi_1(X;u,v)_H$ together with a groupoid structure in $\mathrm{MTH}(k)$. In addition, $\mathcal{O}\big(\pi_1(X;u,v)_H\big)$ has non-negative weights.

It follows that the loop γ_x defines a morphism $\mathbb{Q}(1)_H \to \pi_1(X;u)_H$ of pro-unipotent groups in $\mathrm{MTH}(k)$ whose Betti realization sends $n \in \mathbb{Z}$ to $\gamma_x^n \in \pi_1(X(\mathbb{C});u)$ (the latter morphism does not exist for actual base points!). The de Rham realization of this morphism $T(\Omega) \to \mathcal{O}(\mathbb{G}_a) = T(k)$ is induced by $\mathrm{res}_x : \Omega \to k$.

The case $D = \{0, 1, \infty\}$: One has that Ω has the basis $\omega_0 := \frac{dz}{z}$, $\omega_1 := \frac{dz}{z-1}$. By 0 (resp., 1) denote the tangential point based at 0 (resp., 1) with the tangent vector $\overrightarrow{01}$ (resp., $\overrightarrow{10}$). Thus, *dch* from Talk 2 gives a path in $\pi_1(X;0,1)$. An explicit calculation shows that there is an equality

$$\zeta(\bar{n}) = \int_{dch} \bar{\omega}(\bar{n}) = comp\big(\bar{\omega}(\bar{n})\big)(dch)\,,$$

where $\bar{\omega}(\bar{n})$ is defined as in Talk 2.

Conclusion: MZV's are real periods of an ind mixed Tate Hodge structure over $\mathbb{Q}$ with non-negative weights.

References: [3, §§2.5,3.3], [4, §15], [7, §3], [8, §4]

Thursday. Motivic structure on the fundamental group

8. Tannakian categories

The goal of this talk is first to recall the basic formalism of Tannakian categories (without proofs). Then one shows how an information about the fundamental group of a Tannakian category leads to upper bounds on periods. The case of a semi-product of a pro-unipotent group with $\mathbb{G}_m$ is considered in more detail, which is the main result of the talk.

Basic notions: Definition of a (neutral) Tannakian category $\mathcal{C}$ over a field k, definition of a fiber functor $\omega\colon \mathcal{C} \to \mathrm{Vect}(k)$. Example: $\mathcal{C} = \mathrm{Rep}(G)$ for a linear pro-algebraic group G. Isomorphism scheme $I(\omega,\eta) := \underline{\mathrm{Isom}}_k^{\otimes}(\omega,\eta)$ associated with two fiber functors ω, η. Given an object S in $\mathcal{C}$, one has a canonical k-linear map

$$\omega(S) \otimes_k \eta(S)^\vee \to \mathcal{O}\big(I(\omega,\eta)\big)\,.$$

If $\mathcal{C}$ is tensor generated by S, then $I(\omega,\eta)$ is a closed subvariety in $\underline{\mathrm{Isom}}_k(\omega(S),\eta(S))$. The fundamental group $G_\omega := I(\omega,\omega)$ of $(\mathcal{C},\omega)$. The scheme $I(\omega,\eta)$ is a right torsor under G_ω and a left torsor under G_η. Main theorem: $(\mathcal{C},\omega)$ is equivalent to $\mathrm{Rep}(G_\omega)$ with the forgetful functor.

Examples: Graded vector spaces and the group $\mathbb{G}_m$. Local systems and their fibers. Unipotent completion of an abstract group. Mixed (Tate) Hodge structures over $\mathbb{Q}$ with two fiber functors ω_{dR},ω_B to $\mathrm{Vect}(\mathbb{Q})$ and a point $comp \in I(\omega_{dR},\omega_B)(\mathbb{C})$.

Upper bounds on periods: Let $k \subset K$ be a field extension, p be a K-point on $I(\omega,\eta)$, and let S be an (ind-)object in $\mathcal{C}$. Put $P \subset K$ to be the k-subspace generated by "periods" of type $\langle \alpha, p^\vee(\beta)\rangle$, where $\alpha \in \omega(S)$, $\beta \in \eta(S)^\vee$. Then P is a quotient of the subspace in $\mathcal{O}\big(I(\omega,\eta)\big)$ generated by the image of $\omega(S)\otimes_k \eta(S)^\vee$. Besides, if $\mathcal{O}\big(I(\omega,\eta)\big)$ is generated as a k-algebra by periods of S, then $\mathcal{C}$ is tensor generated by S.

Further, let $c\colon K \to K$ be a field involution over k and assume that $\mathrm{char}(k) \neq 2$. Suppose that c extends to an involution of $I(\omega,\eta)$ over k that commutes with the morphism $p\colon \mathrm{Spec}(K) \to I(\omega,\eta)$ (such an extension may be not unique). Then P^c is a subquotient of $\mathcal{O}\big(I(\omega,\eta)\big)^c$.

Pro-unipotent group: Until the end of the talk we assume that $\mathrm{char}(k) = 0$. The fundamental group G_ω of $(\mathcal{C},\omega)$ is pro-unipotent iff every object in $\mathcal{C}$ has a filtration with quotients being the unit objet $\mathbb{1}$. Suppose that the k-vector space $V := \mathrm{Ext}^1_{\mathcal{C}}(\mathbb{1},\mathbb{1})$ is of finite dimension r and $\mathrm{Ext}^2_{\mathcal{C}}(\mathbb{1},\mathbb{1}) = 0$. Then there is an isomorphism of Hopf algebras $\mathcal{O}(G_\omega) \simeq T(V)$, that is, G_ω is the pro-unipotent completion of a free group on r generators. To show this, one first uses the isomorphisms

$$\mathrm{Ext}^1_{\mathcal{C}}(\mathbb{1},\mathbb{1}) \simeq H^1(G_\omega,k) \simeq \mathrm{Hom}(G_\omega,\mathbb{G}_a)$$

in order to construct a surjective morphism $\mathrm{Spec}\,T(V) \to G_\omega$. Then one uses the interpretation of $\mathrm{Ext}^2_{\mathcal{C}}(\mathbb{1},\mathbb{1}) \simeq H^2(G_\omega,k)$ in terms of extensions of G_ω by $\mathbb{G}_a$.

Semi-product of a pro-unipotent group with $\mathbb{G}_m$: Let L be a rank one object in $\mathcal{C}$ and put $L^{-1} := L^\vee$. Assume that every object S in $\mathcal{C}$ has an increasing filtration w_nS, $n \in \mathbb{Z}$, whose n'th adjoint quotient is a direct sum of several copies of $L^{\otimes(-n)}$. Suppose that the filtration w is exact, respects morphisms in $\mathcal{C}$, and the tensor structure. In particular, $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, L^{\otimes n}) = 0$ for $n \neq 0$ and $\mathrm{Ext}^1_{\mathcal{C}}(\mathbb{1}, L^{\otimes n}) = 0$ for $n \leqslant 0$. Then one has a canonical fiber functor over k

$$\omega\colon S \mapsto \bigoplus_n \mathrm{Hom}_{\mathcal{C}}(L^{\otimes(-n)}, \mathrm{gr}^w_n S)\,.$$

The corresponding fundamental group has the form $G_\omega \simeq \mathbb{G}_m \ltimes U$, where U is a pro-unipotent group. With this identification L corresponds to the representation of G_ω that factors through the tautological representation of $\mathbb{G}_m$ and ω corresponds to the restriction to the subgroup $\mathbb{G}_m \subset G_\omega$. Action of $\mathbb{G}_m$ on U by conjugation defines a grading on the Hopf algebra $\mathcal{O}(U)$ and $\mathcal{C}$ is equivalent to the category of graded comodules over $\mathcal{O}(U)$. In order to be coherent with the filtration w, we say that a representation of $\mathbb{G}_m$ has degree n if an element $\lambda \in \mathbb{G}_m$ acts as multiplication by λ^{-n} (in particular, L corresponds to the degree -1 trivial comodule over $\mathcal{O}(U)$).

Example: for $k \subset \mathbb{C}$, put $\mathcal{C} = \mathrm{MTH}(k)$ with $L = \mathbb{Q}(1)_H$ and $w_n = W_{2n}$ (the canonical fiber functor ω is canonically isomorphic to ω_{dR}). It follows that $\mathrm{MTH}(k)$ is equivalent to $\mathrm{Rep}(G_H)$, where $G_H \simeq \mathbb{G}_m \ltimes U_H$ with U_H being a pro-unipotent over k.

Suppose that for any n, the k-vector space $V_n := \mathrm{Ext}^1_{\mathcal{C}}(\mathbb{1}, L^{\otimes n})$ has finite dimension r_n and $\mathrm{Ext}^2_{\mathcal{C}}(\mathbb{1}, L^{\otimes n}) = 0$ (in particular, $r_n = 0$ for $n \leqslant 0$). Then there is an isomorphism of graded Hopf algebras

$$\mathcal{O}(U) \simeq T\big(\textstyle\bigoplus_{n>0} V_n\big)\,,$$

where V_n has degree n (this is a graded version of the argument from the previous section).

Periods in the pro-unipotenet case: We use notation and assumptions from previous sections. In particular, ω is the canonical fiber functor over k and η is an arbitrary one. Suppose that $\mathrm{gr}^n_W S = 0$ for $n < 0$. Also, suppose that the involution $c\colon I(\omega,\eta) \to I(\omega,\eta)$ is equal to the action of an order two element ϵ in G_η with respect to the left G_η-torsor structure on $I(\omega,\eta)$. Filtration on S defines a filtration on $\omega(S)$, which, in turn, defines a filtration on P (we put a trivial filtration on $\eta(S)^\vee$). Consider the increasing filtration on the graded k-vector space

$$k[t^2] \otimes_k T\big(\textstyle\bigoplus_{n>0} V_n\big)\,,$$

where t has degree 1. Then the filtered k-vector space P^c is isomorphic to a (strict) subquotient of $k[t^2] \otimes_k T\big(\bigoplus_{n>0} V_n\big)$.

The proof is as follows. First, any G_ω-torsor is trivial, because $H^1(k, \mathbb{G}_a)$ and $H^1(k, \mathbb{G}_m)$ are trivial. Thus, we may assume that $\eta = \omega$. The morphism

$$\mathbb{G}_m \times U \to G_\omega, \quad (a,u) \mapsto a \cdot u$$

defines an isomorphism of graded algebras

$$\mathcal{O}(G_\omega) \simeq k[t, t^{-1}] \otimes_k \mathcal{O}(U)\,,$$

where the grading on $\mathcal{O}(G_\omega)$ is induced by right translations by $\mathbb{G}_m \subset G_\omega$. Since $\mathrm{gr}^n_W S = 0$ for $n < 0$, the action of G_ω on $\omega(S)$ factors through the monoid $\mathbb{A}^1 \ltimes U$. Hence, the image of $\omega(S) \otimes \eta(S)^\vee$ is contained in the subalgebra $k[t] \otimes_k \mathcal{O}(U)$. On the other hand, as any order two element in $G_\eta = G_\omega$, ϵ is conjugate to the element $-1 \in \mathbb{G}_m \subset G_\omega$, while $\mathcal{O}(\{-1\}\backslash G_\omega) = k[t^2, t^{-2}] \otimes_k \mathcal{O}(U)$. Since translations in G_ω preserve the subalgebra $\mathcal{O}(\mathbb{A}^1 \ltimes U)$ in $\mathcal{O}(G_\omega)$, this finishes the proof.

References: [4, §8.10], [5, §1], [6, §§A.13, A.14], [7, §§5.2, 5.5, 5.7], [12, §1]

9. Mixed Tate motives over $\mathbb{Z}$

This talk covers, almost without proofs, some foundational material about motives. The goal is to explain roughly the construction of the Tannakian category of mixed Tate motives over $\mathbb{Z}$. The main result from Talk 8 is applied in order to get an upper bound on real periods of mixed Tate motives over $\mathbb{Z}$.

Triangulated category of motives: We consider motives only with rational coefficients. Let $SmCor(k)$ be the category with objects being smooth varieties over a field k and morphisms being $\mathbb{Q}$-linear combinations of finite correspondences. This is an additive category and one can consider its category of bounded complexes up to homotopy $K^b\big(SmCor(k)\big)$. Then $\mathrm{DM}^{\mathit{eff}}(k)$ is obtained as an idempotent completion of the Verdier localization of $K^b\big(SmCor(k)\big)$ by two types of relations: Mayer–Vietoris for Zariski topology and $\mathbb{A}^1$-invariance. Examples: object $M(X)$ defined by a smooth variety X, $\mathbb{Q}(0) := M(\mathrm{Spec}(k))$, the Tate motive $\mathbb{Q}(1)$, the decomposition $M(\mathbb{P}^1) \simeq \mathbb{Q}(0) \oplus \mathbb{Q}(1)[2]$. One formally inverts the positive Tate twists to get $\mathrm{DM}(k)$. There are triangulated and tensor structures on $\mathrm{DM}(k)$ and rigidity holds: for any motive M in $\mathrm{DM}(k)$, there is a dual motive $M^\vee$. Example: $M(X)^\vee$ is an algebra object.

Realizations: De Rham, Betti, and étale realization functors, which send $M(X)$ to the homology of X (dual to cohomology). The mixed Hodge realization functor $\omega_H \colon \mathrm{DM}(k) \to D^b\big(\mathrm{MH}(k)\big)$. Example: realizations of $\mathbb{Q}(1)$.

Mixed Tate motives: Morphisms in $\mathrm{DM}(k)$ are related to K-groups: $\mathrm{Hom}_{\mathrm{DM}(k)}(M(X), \mathbb{Q}(n)[i]) = K_{2n-i}(X)^{(n)}_{\mathbb{Q}}$. Example:

$$\mathrm{Hom}_{\mathrm{DM}(k)}(\mathbb{Q}(0), \mathbb{Q}(1)[1]) = k^* \otimes_{\mathbb{Z}} \mathbb{Q}\,.$$

Some people believe in the existence of an abelian tensor category $\mathrm{MM}(k)$ of mixed motives over k with $D^b\big(\mathrm{MM}(k)\big)$ being equivalent to $\mathrm{DM}(k)$. For a smooth variety X, its motive $M(X)$ is expected to be in $D^{\leqslant 0}\big(\mathrm{MM}(k)\big)$. The Beilinson–Soulé conjecture: $\mathrm{Hom}_{\mathrm{DM}(k)}(M(X), \mathbb{Q}(n)[i]) = 0$ for $i < 0$ and all n.

Let $\mathrm{DTM}(k)$ be the minimal full triangulated subcategory in $\mathrm{DM}(k)$ that contains all $\mathbb{Q}(n)[i]$. Let $\mathrm{MTM}(k)$ be the full subcategory in $\mathrm{DM}(k)$ formed by iterated extensions of all $\mathbb{Q}(n)$ (this is a rigid tensor category). Assume the Beilinson–Soulé conjecture holds for $X = \mathrm{Spec}(k)$ (this amounts to an understanding of K-groups of k). Then $\mathrm{MTM}(k)$ is abelian (moreover, it is a heart of a t-structure on $\mathrm{DTM}(k)$). For all M, N in $\mathrm{MTM}(k)$, one has

$$\mathrm{Ext}^1_{\mathrm{MTM}(k)}(M,N) = \mathrm{Hom}_{\mathrm{DM}(k)}(M, N[1]), \quad \mathrm{Ext}^2_{\mathrm{MTM}(k)}(M,N) \subset \mathrm{Hom}_{\mathrm{DM}(k)}(M, N[2])\,.$$

There is an exact tensor functorial increasing weight filtration W_n on mixed Tate motives whose $2n$'the adjoint quotients are direct sums of copies of $\mathbb{Q}(-n)$ and whose odd adjoint quotients are trivial. Given $k \subset \mathbb{C}$, one has the mixed Hodge realization functor $\omega_H \colon \mathrm{MTM}(k) \to \mathrm{MTH}(k)$, which respects the weight filtrations. Example: the Kummer mixed Tate motive defined by an element from $k^* \to \mathrm{Ext}^1_{\mathrm{MTM}(k)}(\mathbb{Q}(0), \mathbb{Q}(1))$, its mixed Hodge and étale realizations.

Borel's result: Borel has computed explicitly K-groups of number fields. For $k=\mathbb{Q}$, this gives the following:

$$\mathrm{Hom}_{\mathrm{DM}(\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n)[i])=0 \quad \text{for } i\neq 0,1\,,$$

$$\mathrm{Hom}_{\mathrm{DM}(\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n)[1])=\begin{cases}\mathbb{Q}^{\times}\otimes_{\mathbb{Z}}\mathbb{Q} & \text{if} \quad n=1,\\ 0 & \text{if} \quad n \text{ is even or negative},\\ \mathbb{Q} & \text{if} \quad n\geq 3 \text{ and odd}.\end{cases}$$

Therefore, the Beilinson–Soulé conjecture holds for $\mathrm{Spec}(\mathbb{Q})$ and the category $\mathrm{MTM}(\mathbb{Q})$ is abelian.

In addition, it follows from Borel's arguments that ω_H induces injective maps

$$\mathrm{Ext}^1_{\mathrm{MTM}(\mathbb{Q})}(\mathbb{Q}(0),\mathbb{Q}(n))\hookrightarrow \mathrm{Ext}^1_{\mathrm{MTH}(\mathbb{Q})}(\mathbb{Q}(0)_H,\mathbb{Q}(n)_H)\,.$$

Mixed Tate motives over $\mathbb{Z}$: The category $\mathrm{MTM}(\mathbb{Z})$ of mixed Tate motives over $\mathbb{Z}$ is a full subcategory of $\mathrm{MTM}(\mathbb{Q})$ that consists of all objects whose étale realization is unramified at all primes. Example: non-trivial Kummer motives are not from $\mathrm{MTM}(\mathbb{Z})$. Equivalent description: M is from $\mathrm{MTM}(\mathbb{Z})$ iff $W_{-n}/W_{-n-2}(M)$ is a direct sum of $\mathbb{Q}(n)$'s and $\mathbb{Q}(n+1)$'s for all $n\in\mathbb{Z}$. This has the effect of killing $\mathrm{Ext}^1(\mathbb{Q}(0),\mathbb{Q}(1))$, while keeping the same $\mathrm{Ext}^1(\mathbb{Q}(0),\mathbb{Q}(n))$ as in $\mathrm{MTM}(\mathbb{Q})$ for the other n's and keeping the vanishing $\mathrm{Ext}^2(\mathbb{Q}(0),\mathbb{Q}(n))=0$ for all n. If the motive $M(X)$ of a variety X over $\mathbb{Q}$ is from $D^b\big(\mathrm{MTM}(\mathbb{Q})\big)$ and X has good reduction over all primes (which is quite rare), then $M(X)$ is actually from $D^b\big(\mathrm{MTM}(\mathbb{Z})\big)$.

The category $\mathrm{MTM}(\mathbb{Z})$ is neutral Tannakian. The de Rham and Betti realizations give fiber functors over $\mathbb{Q}$, denoted by ω_{dR} and ω_B, respectively. One sees that $\mathrm{MTM}(\mathbb{Z})$ satisfies the condition from Talk 8 with $L=\mathbb{Q}(1)$ and $w_n=W_{2n}$ (the canonical fiber functor ω is canonically isomorphic to ω_{dR}). It follows that $\mathrm{MTM}(\mathbb{Z})$ is equivalent to $\mathrm{Rep}(G_M)$, where $G_M\simeq\mathbb{G}_m\ltimes U_M$ with U_M being a pro-unipotent group over $\mathbb{Q}$.

The functor ω_H corresponds to a morphism of pro-algebraic groups $G_H\to G_M$. Since ω_H is injective on Ext^1's, this morphism is surjective (one should use that U_M and U_G are pro-unipotent). Therefore, the functor ω_H is fully faithful and its image is essentially closed under taking subquotients.

Upper bound on periods: For any variety X over $\mathbb{Q}$, one has the complex conjugation on $X(\mathbb{C})$. This defines an automorphism of the fiber functor ω_B, that is, an element ϵ of order two in the corresponding fundamental group G_{ω_B} over $\mathbb{Q}$. The comparison isomorphism defines a $\mathbb{C}$-point *comp* on $I(\omega_{dR},\omega_B)$. The complex conjugation $c\colon\mathbb{C}\to\mathbb{C}$ commutes via *comp* with the involution of $I(\omega_{dR},\omega_B)$ given by the action of ϵ (this can be proved by an explicit local calculation with differential forms). Thus, by results from Talk 8, the filtered subspace P^c in $\mathbb{R}$ generated by all real periods of mixed Tate motives over $\mathbb{Z}$ with non-negative weights is a (strict) subquotient of the graded $\mathbb{Q}$-vector space

$$\mathbb{Q}[t^2]\otimes_{\mathbb{Q}} T\big(\textstyle\bigoplus_{n>0}\mathbb{Q}\cdot e_{2n+1}\big)\,.$$

In particular, if e_n is the dimension of the n-th adjoint quotient of P^c, then $e_n\leqslant e_{n-2}+e_{n-3}$ (write down explicitly an upper bound on the corresponding Poincaré series).

References: [6, §1], [7, §§5.1, 5.2, 5.4]

10. Motivic fundamental group

In this talk one constructs for special varieties a pro-unipotent group in $\mathrm{MTM}(\mathbb{Z})$ whose realization is the mixed Hodge fundamental group from Talk 6. The case of $\mathbb{P}^1$ without three points is considered in more detail, which leads to Goncharov–Terasoma's theorem with the help of results from Talk 9.

Motivic bar complex: The idea is that every "complex of smooth varieties" over k gives an object in $\mathrm{DM}(k)$. Let X be a smooth variety over k, $a, b \in X(k)$. One replaces in the bar complex (see Talk 4) the dg-algebra $A^{\bullet}_M$ by the algebra object $M(X)^{\vee}$ in $\mathrm{DM}(k)$. Since the differential, the product, and the coproduct structures on the bar complex have a geometric origin, this defines an algebra ind-object $B(X; a, b)$ in $\mathrm{DM}(k)$ together with the coproduct structure. Equivalently, $B(X; a, b)$ is the dual to the pro-motive given by the cosimplicial path variety $P^{\bullet}(X; a, b)$.

Mixed Tate case: Assume that the Beilinson–Soulé conjecture is true for $\mathrm{Spec}(k)$, whence we have an abelian category $\mathrm{MTM}(k)$ being a heart of a t-structure on $\mathrm{DTM}(k)$. Suppose that $M(X)$ is from $\mathrm{DTM}(k) \subset \mathrm{DM}(k)$. Then $B(X; a, b)$ is an ind-object from $\mathrm{DTM}(k)$. Taking H^0 with respect to the t-structure, one obtains an algebra ind-object in $\mathrm{MTM}(k)$. This defines mixed Tate motivic affine schemes $\pi_1(X; a, b)_M$ together with a groupoid structure in $\mathrm{MTM}(k)$. By construction, if $k \subset \mathbb{C}$, then the mixed Hodge realization of $\pi_1(X; a, b)_M$ is $\pi_1(X; a, b)_H$.

Aside on mixed Hodge tangential fundamental group: Consider $X = \mathbb{P}^1 \smallsetminus D$ as in Talk 7, points $a \in X(k)$, $\infty \neq y \in D(k)$, and a non-zero tangent vector $v \in T_y\mathbb{P}^1$. Assume that $\infty \in D$. The embedding $X \subset \mathbb{P}^1 \smallsetminus \{y, \infty\}$ induces a morphism between affine schemes in $\mathrm{MTH}(k)$

$$\pi_1(X; a, v)_H \to \pi_1(\mathbb{P}^1 \smallsetminus \{y, \infty\}; a, v)_H \,.$$

Consider the isomorphism $\varphi\colon T_y\mathbb{P}^1 \simeq \mathbb{P}^1 \smallsetminus \{\infty\}$ that sends 0 to y and whose differential is the identity. It follows from Talk 7 that there is an isomorphism

$$\pi_1(\mathbb{P}^1 \smallsetminus \{y, \infty\}; a, v)_H \simeq \pi_1(T_y\mathbb{P}^1 \smallsetminus \{0\}; \varphi^{-1}(a), v)_H \,,$$

where the second affine scheme in $\mathrm{MTH}(k)$ corresponds to paths between actual points and is a torsor under $\mathbb{Q}(1)_H$.

On the other hand, one has a morphism between affine schemes in $\mathrm{MTH}(k)$

$$\pi_1(X; a, v)_H \times \mathbb{Q}(1)_H \to \pi_1(X; a)_H$$

whose Betti realization sends (γ, n) to $\gamma \circ \gamma_y^n \circ \gamma^{-1} \in \pi_1(X(\mathbb{C}); a)$, $n \in \mathbb{Z}$. This induces the morphism

$$\pi_1(X; a, v)_H/\mathbb{Q}(1)_H \to \mathrm{Hom}\big(\mathbb{Q}(1)_H, \mathrm{Lie}\big(\pi_1(X; a)_H\big)\big) =: \mathrm{Lie}\big(\pi_1(X; a)_H\big)(-1) \,,$$

where the action of $\mathbb{Q}(1)_H$ on $\pi_1(X; a, v)_H$ is through the morphism $\mathbb{Q}(1)_H \to \pi_1(X; v)_H$. Taking the de Rham realization and using a general result about free pro-unipotent groups (the proof is not required), one shows that the latter morphism is a closed embedding.

All together, this leads to a closed embedding between affine schemes in $\mathrm{MTH}(k)$

$$\pi_1(X; a, v)_H \hookrightarrow \pi_1(T_y\mathbb{P}^1 \smallsetminus \{0\}; \varphi^{-1}(a), v)_H \times \mathrm{Lie}\big(\pi_1(X; a)_H\big)(-1) \,.$$

Motivic tangential fundamental group: Both the affine scheme $\pi_1(T_y\mathbb{P}^1 \smallsetminus \{0\}; \varphi^{-1}(a), v)_H$ and the vectorial space $\mathrm{Lie}\big(\pi_1(X;a)_H\big)(-1)$ in $\mathrm{MTH}(k)$ are in the image of the functor ω_H. By the properties of this functor from Talk 9, we obtain a mixed Tate motivic affine scheme $\pi_1(X;a,v)_M$ with an action of $\pi_1(X;a)_M$. Further, taking products of torsors leads to mixed Tate motivic affine schemes $\pi_1(X;u,v)_M$ together with a groupoid structure in $\mathrm{MTM}(k)$. In addition, $\mathcal{O}\big(\pi_1(X;u,v)_M\big)$ has non-negative weights. Also, we have a morphism between motivic pro-unipotent groups $\mathbb{Q}(1) \to \pi_1(X;u)_M$ whose mixed Hodge realization is the morphism $\mathbb{Q}(1)_H \to \pi_1(X;u)_H$.

The case of $X = \mathbb{P}^1 \smallsetminus \{0,1,\infty\}$: We obtain an ind mixed Tate motive $\mathcal{O}\big(\pi_1(X;0,1)_M\big)$ with non-negative weights such that MZV's appear as (some of) its real periods. An upper bound on periods from Talk 9 immediately implies Goncharov–Terasoma's theorem.

References: [6, §§3.12, 4.4], [10]

Friday. Brown's proof

11. Zagier's theorem

The theorem is proved in [16] (see also [11]).

12. The proof. Part 1

Up to Joseph, Sergey, and Sergey.

13. The proof. Part 2

Up to Joseph, Sergey, and Sergey.

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