OVERCONVERGENT MODULAR FORMS AND THE ARTIN CONJECTURE

The first goal of this summer school is to understand and prove the following theorem.

Théorème 0.1. *Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ be irreducible and odd with projective image A_5. Assume that ρ is unramified at 5. Then there exists a cuspidal weight one form f such that $L(f, s) = L(\rho, s)$. In particular, ρ satisfies the Artin conjecture.*

This theorem is due to Taylor, Buzzard and Shepherd-Barron and was initiated just after the proof of Fermat Last Theorem by Wiles. We will see indeed that many ingredients of Wiles are used in the proof. Remark also that the case of projective image A_5 was the only unknown case in dimension two.

This theorem is also due to Khare and Wintenberger as a corollary of their proof of Serre's modularity conjecture (so latter with a different proof). They even managed to remove the unramifiedness hypothesis at 5.

Today one can adapt the proof of Taylor and friends to arbitrary totally real fields, and remove by the same occasion the unramifiedness assumption at 5.

The other goal of the summer school is to introduce p-adic modular forms, and to prove various classicality theorems for such p-adic modular forms. Here a classicity theorem is a statement saying that a particular p-adic modular form with good properties is in fact a usual modular form, perhaps just with a q-expansion in $\bar{\mathbb{Q}}_p[[q]]$ and not in $\mathbb{C}[[q]]$.

The link between both subjects will be a classicality result for 5-adic modular forms of weight one (we will try to understand why 5 plays such a strange role). Indeed, methods of Wiles will allow us to construct an object f associated to ρ as in the theorem, but f will be a 5-adic modular form of weight one, and perhaps not a classical form. To know the holomorphy of $L(f, s)$ (and in fact already to define $L(f, s)$ as a function of a complex variable s) one would need to prove the classicality of f.

All talks, including the introduction one, will last 1h30. Except in the introduction talk, every statement should be proven. You should use the notations of the following text, and not those of the references provided, because this will provide some uniformity between the different talks. Speakers can exchange material between adjacent talks. If you have any question on your talks (mathematical questions, references,...) you should contact me at
benoit.stroh@gmail.com

1. Sunday : Introduction

1.1. Talk 1 : Artin conjecture.

No proof in this talk.

Let F be a number field, $\bar{F}$ an algebraic closure and $G_F = \mathrm{Gal}(\bar{F}/F)$ the absolute Galois group. Introduce the complex Galois representations $\rho : G_F \to \mathrm{GL}_n(\mathbb{C})$. Explain why ρ is continuous if and only if it has finite image.

Introduce the L-function $L(\rho,s)=\prod_{\mathfrak{p}}\det(1-\rho(\mathrm{Frob}_{\mathfrak{p}})N\mathfrak{p}^{-s}:(\mathbb{C}^n)^{I_{\mathfrak{p}}})^{-1}$ where $\mathfrak{p}$ run through the primes ideals of $\mathcal{O}_F$ and $I_{\mathfrak{p}}\subset G_F$ is the inertia subgroup of $\mathfrak{p}$. Give the region of convergence of $L(\rho,s)$.

State Artin conjecture : if ρ is irreducible non trivial, $L(\rho,s)$ has an holomorphic continuation to $\mathbb{C}$. Just say (without any recollection on class field theory) that Artin conjecture is true if $\dim(\rho)=1$. Moreover, Brauer's argument shows that $L(\rho,s)$ has meromorphic continuation to $\mathbb{C}$ and satisfies a functional equation for any ρ.

Recall quickly what is a modular form of weight $k\geq 1$ and level $\Gamma\subset \mathrm{SL}_2(\mathbb{Z})$ in analytic terms : an holomorphic function on Poincaré half-plane satisfying a functional equation for the action of Γ. If f is a modular form, define $L(f,s)$ which looks like (if $k=1$) an Artin L-function associated to a Galois representation of dimension two. Explain that $L(f,s)$ is known to have analytic continuation to $\mathbb{C}$. Therefore to prove that $\rho: G_F\to \mathrm{GL}_2(\mathbb{C})$ satisfy Artin's conjecture, it is enough to prove that there exists a weight one form f such that $L(f,s)=L(\rho,s)$.

Then focus on the case where $\dim(\rho)=2$. Recall the classification of irreducible finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$: dihedral, A_4, S_4, A_5. Except A_5, all are solvable. Give the following theorem of Langlands and Tunnell : in all cases of solvable image in dimension two, Artin conjecture is true. Therefore it is enough to focus on the remaining A_5-case.

Now restrict to $F=\mathbb{Q}$. Explain the definition of oddness for $\rho: G_{\mathbb{Q}}\to \mathrm{GL}_2(\mathbb{C})$: the determinant of $\rho(c)$ is -1 for any complex conjugation $c\in G_{\mathbb{Q}}$. State the main theorem of this summer school, which is due to Taylor, Buzzard and Shepherd-Barron, and which was given at the beginning of this text.

References : [N, ch.VII.10,11,12], [DS1, ch.5], [C]

2. Monday : modular forms and Galois representations

2.1. Talk 2 : Modular curves and forms. Let $N\geq 5$ be an integer. Introduce the modular curve X of level $\Gamma_1(N)$ over $\mathrm{Spec}(\mathbb{Z}[1/N])$ as the moduli space of elliptic curves with a point of order N. Explain briefly why it is representable (follow the argument of T. Saito; we used that $N\geq 5$ to be sure that objects have no non-trivial automorphisms). In this way, we get a smooth affine curve of $\mathrm{Spec}(\mathbb{Z}[1/N])$. Explain its canonical compactification $\bar{X}$, which is a smooth projective curve. The universal elliptic curve extends to a smooth but non proper group scheme E over the compactification $\bar{X}$: over the cusps, its fiber is the multiplicative group $\mathbb{G}_m$. Introduce the sheaf

$$\omega=e^*\omega^1_{E/\bar{X}}$$

where $e:\bar{X}\to E$ is the unit section. Define modular forms of weight k and level $\Gamma_1(N)$ as global sections of the line bundle ω^k on $\bar{X}$. Make the link with the analytic definition of modular forms (follows T. Saito). Advantage of the geometric definition of modular forms : we have a canonical integral structure of $\mathbb{Z}[1/N]$-module on the complex vector space of modular forms.

Let p be a prime not dividing N. Introduce $X_0(p)\to \mathrm{Spec}(\mathbb{Z}[1/Np])$ as the moduli space of elliptic curves E plus a point of order N plus a subgroup $H\subset E[p]$ of rank p

in the p-torsion of E. There are two maps $\pi_1, \pi_2 : X_0(p) \to X$ given in modular terms by $(E, H) \mapsto E$ and E/H. This gives rise to $(\pi_1, \pi_2) : X_0(p) \hookrightarrow X \times X$ which can be seen as an algebraic correspondence on the curve X. Everything extends to $\bar{X}$. Then introduce Hecke operators geometrically as Saito, 2.6.

Recall what are ordinary and supersingular elliptic curves over $\mathrm{Spec}(\mathbb{F}_p)$. Introduce the integral model of $X_0(p)$ over $\mathrm{Spec}(\mathbb{Z}[1/N])$. By definition it parametrizes E an elliptic curve, a point P of order N and a finite flat subgroup $H \subset E[p]$ (note : finite flat subgroups will be discussed in more details in the talk 6). Same for the compactification $\bar{X}_0(p)$. The scheme $\bar{X}_0(p)$ is not smooth over $\mathrm{Spec}(\mathbb{F}_p)$ but has two irreducible components, which are isomorphic to two $\mathbb{P}^1_{\mathbb{F}_p}$ meeting transversally. The meeting points corresponds to (the finite number) of supersingular elliptic curves. Generically on the first component, E is ordinary and H is etale. Generically on the second component, E is ordinary and $H \simeq \mu_p$ is multiplicative. Explain that in great details, with many pictures. It will be a very important ingredient in all the other talks. References are Saito, Deligne-Rapoport (with a complicated stacky language), Diamond-Shurman (which avoids the algebraic geometry as much as possible) or Katz, 1.13.

References : [S2], [DS1, ch.7], [DR, ch.V], [K1].

2.2. Talk 3 : Galois representations for high weight modular forms. Define what are eigenforms : they are eigen for all Hecke operators T_p where p does not divide the level N. To such a cuspidal eigenform f and to any prime number ℓ we want to associate a continuous Galois representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_\ell)$ (with infinite image) such that $L(f, s) = L(\rho_f, s)$ where $L(\rho_f, s)$ is defined exactly as in the first lecture in the case of finite Galois representations. In particular we want ρ_f to be unramified away from $N\ell$. This full compatibility $L(f, s) = L(\rho_f, s)$ would in fact be very difficult to show at primes dividing $N\ell$, and we will just show equality between all factors of the Euler products at primes not dividing $N\ell$. This amounts to show that $\mathrm{tr}(\rho_f(\mathrm{Frob}_p)) = a_p$ for all $p \nmid N\ell$ where a_p is the eigenvalue of T_p on f.

First focus on weight two modular forms. Follow Saito, par.3. State first Kodaira-Spencer isomorphism $\Omega^1_X = \omega^2(-D)$ where ω is the line bundle of weight one forms introduced in the previous talk and $D \hookrightarrow X$ is the set of cusps. Therefore Ω^1_X is the sheaf of weight two cuspidal forms. The Kodaira-Spencer isomorphism follows from classical deformation theory. Then use (complex) Hodge theory to compute $H^1(\bar{X}(\mathbb{C}), \mathbb{C})$ as the sum of two copies of $S_2(\Gamma_1(N))$, the $\mathbb{C}$-vector space of weight two cusp forms of level $\Gamma_1(N)$. This is the so-called Eichler-Shimura isomorphism.

Then give the construction of ρ_f : it is the part of the etale cohomology $H^1_{\mathrm{et}}(\bar{X} \times \mathrm{Spec}(\bar{\mathbb{Q}}), \bar{\mathbb{Q}}_\ell)$ where T_p acts by multiplication by a_p for all $p \nmid N$, where $T_p(f) = a_p \cdot f$. Note that the etale cohomology can be (artificially) replaced by the Tate module of the Jacobian of $\bar{X}$. Suggestion : do not spend time recalling what is etale cohomology. Eichler-Shimura isomorphism shows that ρ_f is indeed of dimension two. Finally prove the congruence relation as in Saito, 3.4 building on the results of the previous talk.

Explain what need to be done in weight $k \geq 3$: the etale cohomology group should be replaced by $H^1_{\text{para}}(\bar{X} \times \text{Spec}(\bar{\mathbb{Q}}), j_* \text{Sym}^{k-2} T_\ell E)$ which is isomorphic to

$$\text{Im}\left(H^1_c(X \times \text{Spec}(\bar{\mathbb{Q}}), \text{Sym}^{k-2} T_\ell E) \to H^1(X \times \text{Spec}(\bar{\mathbb{Q}}), \text{Sym}^{k-2} T_\ell E)\right)$$

Here $j : X \hookrightarrow \bar{X}$ is the canonical open embedding and $T_\ell E$ is the etale sheaf on X coming from the Tate module of the universal elliptic curve. The only part which is really more difficult than for $k = 2$ is Eichler-Shimura isomorphism which asserts

$$H^1_{\text{para}}(\bar{X} \times \text{Spec}(\bar{\mathbb{Q}}), j_* \text{Sym}^{k-2} T_\ell E) \simeq S_k(\Gamma_1(N))^2 .$$

This follows from Hodge theory with coefficients. Ask me for scans of a previous version of the text of Saito, far more complete, with every relevant computation.

Explain finally why ρ_f is always odd in the sense that $\det(\rho(c)) = -1$ for all complex conjugation $c \in G_{\mathbb{Q}}$. This follows directly from Hodge theory and Eichler-Shimura isomorphism : the action of $\rho(c)$ can be checked on the level of the natural pure Hodge structure of weight $(0, k-1)$ associated to $V = \iota \circ \rho_f$ where $\iota : \mathbb{C} \xrightarrow{\sim} \bar{\mathbb{Q}}_l$ is an abstract isomorphism. But by Hodge theory this complex conjugation exchanges the spaces $V^{k-1,0}$ and $V^{0,k-1}$ therefore the matrix of $\rho(c)$ is diagonal with coefficients $(1, -1)$ in a good basis.

References : [S2]

2.3. **Talk 4 : Galois representations for weight one modular forms.** Let f be a weight one eigen cuspidal modular form of level $\Gamma_1(N)$. We want to construct $\rho_f : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{C})$ with finite image. It can be shown that such a Galois representation can not be found in the etale cohomology of modular curves. Therefore another strategy must be found. This was done by Deligne and Serre who constructed ρ_f by p-adic interpolation using the previous construction for an infinite family of weights ≥ 2, therefore relying on the results of the previous talk.

Introduce the Hasse invariant H as in Katz, 2.0 and 2.1. Discuss its lifting E_{p-1} if $p \geq 5$. As finally we will only consider $p = 5$ it is not worth spending time on the cases where $p < 5$.

Explain now the method of Deligne and Serre, with some emphasis on the paragraph 6, theorem 6.7 and the famous lemma 6.11, which is now called Deligne-Serre lemma. Give indications on the analytic number theory involved in the construction (paragraph 5, 7 and 8). Do not explain paragraph 9 but if you have time, you can speak of the results explained in the paragraph 4. This results should already have been sketched in the first lecture, but without any proof.

Insist on the oddness : as ρ_f is odd if the weight of f is ≥ 2, this is still true for weight one forms. Cultural remark : even Galois representations of dimension two are still expected to be associated to analytic objects, but this objects will be Maas forms with eigenvalue of the Laplacian 1/4. Maas forms are real analytic functions satisfying a functional equation on Poincaré upper half plane, with contrast to weight one forms which are holomorphic on the Poincaré half-plane. But no one knows how to associate Galois representations to Maas forms today...

The following remark is important : what is difficult in Deligne-Serre is to get a *finite* Galois representation. They get it using analytic number theory and construction of ρ_f

mod p for all primes p. If one just want to construct a continuous $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ without knowing it is finite, where p is fixed, things become easier. One construct ρ_f as a limit of $\rho_{f,n} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}/\mathfrak{m}^n)$ when $n \to \infty$, where $\mathcal{O}$ is a the ring of integers of a finite extension of $\mathbb{Q}_p$ depending on f, and $\mathfrak{m}$ its maximal ideal. One does exactly as in Deligne-Serre, paragraph 6, replacing λ by $\mathfrak{m}^n$. Try to explain this, because it will important for Hida theory.

References : [DS2].

3. Tuesday : p-adic geometry of modular curves

3.1. Talk 5 : Rigid-analytic geometry. The goal of this talk is to explain various constructions in rigid analytic geometry of Tate. Rigid analytic varieties are kind of complex manifolds, but over p-adic fields : holomorphic functions are replaced by some converging power series. Tate geometry has important advantages : it allows us to speak about open and closed balls into the rigid varieties associated to a scheme over $\mathbb{Q}_p$. This will be essential to define p-adic modular forms in a next talk. This theory has also some drawbacks : a complicated topological interpretation (a Grothendieck topology and not a normal topology ; also in Tate geometry we speak frequently of « quasi-compact open subset » which are morally both open and compact, which is strange). Those drawbacks can be corrected by the use of Berkovich analytic geometry, but this will not be important for us this week.The standard reference for rigid geometry is Berthelot.

Begin with the definition of Tate algebra. Explain then what a rigid variety following Berthelot, 0.1. Explain what is a quasi-compact rigid variety, and explain why holomorphic functions (in the p-adic sense) are bounded on quasi-compact varieties. Gives examples of rigid varieties : closed balls, open balls, locus where $|f| \geq 1$, locus where $|f| < 1$ where $f \in \mathbb{Q}_p < X, Y, Z, \cdots >$ is in a Tate polynomial algebra.

Explain paragraph 0.2 of Berthelot. Give also many examples. You can for instance see what happen when you complete formally the affine line $\mathbb{A}^1_{\mathbb{Z}_p}$ along its special fiber $\mathbb{A}^1_{\mathbb{F}_p}$. Which rigid variety do you get as its rigid generic fiber in the sense of Berthelot ? And if you complete $\mathbb{A}^1_{\mathbb{Z}_p}$ along the origin $0_{\mathbb{F}_p}$ of the special fiber ? Also explain the specialization morphism of Berthelot.

Do not speak of the paragraph 0.3. It will not be relevant for us.

Explain 1.1 and 1.2 of Berthelot : definition and properties of tubes and their strict neighborhoods. Gives a lot of examples, writing explicit equations for your rigid varieties. Don't explain 1.3 of Berthelot.

Finally state and prove proposition 2.2.4 of [PS1]. It will be useful to construct the canonical subgroup in another talk.

References : [PS2], [B].

3.2. Talk 6 : Rigid-analytic modular curves. Rigid-analytic modular curves are the basic geometric objects on which the theory of p-adic modular forms is based. Moreover they give nice illustrations to general principles of rigid-analytic geometry.

Begin by a quick definition of finite flat group schemes. Give the main theorem of Oort-Tate : finite flat group schemes of order p over the spectrum of a local $\mathbb{Z}_p$-algebra R are in bijection with equivalences classes of couples $(a, b) \in R^2$ such that $ab = \omega_p$ where $\omega_p \in \mathbb{Z}_p$ is an element of valuation one defined by Oort-Tate. The equivalence classes are given by $(a, b) \simeq (\lambda^{p-1}a, \lambda^{1-p}b)$ where $\lambda \in R^*$. Don't give any indication on the proof but deduce the following : finite flat groups schemes of rank p over $\mathrm{Spec}(\bar{\mathbb{F}}_p)$ are isomorphic to $\mathbb{Z}/p$, μ_p or α_p. Moreover finite flat groups schemes of order p over discrete valuation rings R of mixed characteristic $(0, p)$ are in bijection with $[0, 1]$ intersected with the image $v(R)$ of the valuation. Give the recipe to deduce from this number the reduction of the group schemes over $\bar{\mathbb{F}}_p$.

Following the formalism introduced in the previous talk and the notations of the talk 2, introduce X^{rig}, the rigid generic fiber of the formal completion of $X \to \mathrm{Spec}(\mathbb{Z}_p)$ along its special fiber. Therefore X^{rig} is the moduli space in rigid geometry for elliptic curves with good reduction and with a point of order N. Introduce $X^{\mathrm{rig}}_{\mathrm{ord}}$, the tube of the ordinary locus of the special fiber of X. This is a quasi-compact open of X^{rig} parametrizing elliptic curve with good ordinary reduction. Introduce the variants with the compactification : $\bar{X}^{\mathrm{rig}}$ which is the generic fiber of the completion of $\bar{X}$ along its special fiber, and $\bar{X}^{\mathrm{rig}}_{\mathrm{ord}}$. By convention, the cusps in $\bar{X} \times \mathrm{Spec}(\mathbb{F}_p)$ are in the ordinary locus.

Introduce $X_0(p)^{\mathrm{rig}}$ which is the rigid generic fiber of the formal completion of $X_0(p) \to \mathrm{Spec}(\mathbb{Z}_p)$ along its special fiber. Recall (talk 2) that $X_0(p) \times \mathrm{Spec}(\mathbb{F}_p)$ has two irreducible components which are $\mathbb{P}^1_{\mathbb{F}_p}$ intersecting transversally at supersingular points. On this intersection points, the fiber of the universal finite flat subgroup H is isomorphic (locally for the etale topology) to α_p. On the smooth locus of the first component H is isomorphic to $\mathbb{Z}_p$. On the smooth locus of the second component, it is isomorphic to μ_p. One can therefore define the ordinary-multiplicative locus of $X_0(p) \times \mathrm{Spec}(\mathbb{F}_p)$ as the locus where $H \simeq \mu_p$. Same for $\bar{X}_0(p)$. Introduce finally $X_0(p)^{\mathrm{rig}}_{\mathrm{ord-mul}}$ which is the tube of the ordinary-multiplicative locus. It is a quasi-compact open subset. Same for $\bar{X}_0(p)^{\mathrm{rig}}_{\mathrm{ord-mul}}$. Draw pictures.

Introduce the degree function $\deg : \bar{X}_0(p)^{\mathrm{rig}} \to [0, 1] \cap \mathbb{Q}$ which is the valuation of the Oort-Tate parameter of H. This function is locally the valuation of an analytic function on $\bar{X}_0(p)^{\mathrm{rig}}$. The ordinary-multiplicative locus is $\deg^{-1}(1)$ and so on. Remark : the degree function has been defined in a far greater generality by Fargues but, except for some properties, this general definition will not be so important for us.

Introduce the operator U_p which associates p points to one point on $X_0(p)^{\mathrm{rig}}$. By definition $U_p(E, H)$ is the set of all $(E/L, E[p]/L)$ where $L \subset E[p]$ is a generic supplementary of H. Be careful with the following : if (E, H) corresponds to a point of $X_0(p)^{\mathrm{rig}}$ then by definition the elliptic curve E has good reduction over the ring of integers $\mathcal{O}$ of an extension K of $\mathbb{Q}_p$. The operation of schematic closure defines a bijection between subgroups of $E_K[p]$ and finite flat subgroups of $E[p]$ over $\mathrm{Spec}(\mathcal{O})$. But when we speaks about a supplementary L we really mean $L \subset E[p]$ finite flat over $\mathrm{Spec}(\mathcal{O})$ such that $E_K[p] = L_K \oplus H_K$.

Prove the statements of [P1], paragraph 2 (replacing everywhere abelian varieties by elliptic curves). Remark : you can follow this reference to explain Oort-Tate theory and the degree function.

Références : [OT], [Fa], [P1].

3.3. Talk 7 : Hida theory. Alice Pozzi

Follow [P2] adapting everything to GL_2.

4. Wednesday

4.1. Talk 8 : Residually modular Galois representations. In this talk we will consider the prime number $p = 5$. We will look at a residual Galois representation $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ and show that under some hypothesis, it arises as the reduction modulo p of a ρ_f with f modular. The goal of the talk is to present the main result of [T], but perhaps following [PS1] (replacing everywhere totally real fields by $\mathbb{Q}$).

Begin with the lemma 1.1 of [PS1], which summarizes solvable base change due to Langlands and Tunnell. Let $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}(\bar{\mathbb{F}}_5)$ be an odd Galois representation with projective image A_5. Explain lemma 2.2 of [PS1], which relies on class field theory. Explain lemma 2.3 in detail, following [SBT] (prove what you want in this reference). At this point we found a totally real field F solvable over $\mathbb{Q}$ and an elliptic curve E over F such that $E[3]$ is an irreducible Galois representation and $E[5]$ is isomorphic to $\bar{\rho}$ restricted to G_F up to twist.

Explain how we can deduce from this the modularity of $\bar{\rho}|_{G_F}$ (be careful : this notion of modularity for representations of G_F is related to automorphic representations for $\mathrm{GL}_2(\mathbb{A}_F)$ or to Hilbert modular forms, but not to usual modular forms). For this argue as in [PS1], proposition 2.4, using Kisin's $R = T$ theorem (do not even state this theorem, some talks will speak about analogous theorems latter). This game between 3 and 5 is due to Wiles in his proof of Fermat last theorem. Use finally lemma 1.1 to deduce the modularity of $\bar{\rho}$ as a representation of $G_{\mathbb{Q}}$.

Remark : for $F = \mathbb{Q}$ you can just forget the notion of « ordinairement modulaire » of prop.2.4 : this notion is equivalent to the modularity in the cases at hand.

References : [T], [SBT], [PS1]

5. Thursday : Classicality of overconvergent modular forms

5.1. Talk 9 : Overconvergent modular forms. In this talk, you will define the fundamental notion of overconvergent modular forms, and prove that Hida theory gives rise to overconvergent modular forms.

First define overconvergent modular forms. Use notations introduced in the talk 6. By definition, an overconvergent modular form of weight $k \in \mathbb{Z}$ is a section of the line bundle ω^k defined on a strict neighborhood of $\bar{X}_0(p)^{\mathrm{rig}}_{\mathrm{ord-mul}}$ in $\bar{X}_0(p)^{\mathrm{rig}}$. Therefore it is a section on ω^k defined on $\deg^{-1}([1-\varepsilon, 1])$ for $\varepsilon > 0$ because such neighborhood are cofinal in the set of all strict neighborhoods. Here I defined more precisely overconvergent modular forms of level $\Gamma_1(N)$ outside p and level $\Gamma_0(p)$ at p. You will find sometimes variants with $\Gamma_0(p)$ replaced by $\Gamma_1(p^n)$ but this is useless for us.

For all that you can follow Kassaei par 3.1 and 3.2 but try to formulate things with the degree function instead of the precise bounds with E_{p-1} used by Kassaei : they are unuseful.

Explain the action of the Hecke operators T_ℓ and U_p on the space of overconvergent modular forms. The most interesting is U_p and you will use the fact that for all $\varepsilon \in]0, 1[$, there exist $\eta < \varepsilon$ such that

$$U_p\left(\deg^{-1}([1-\varepsilon])\right) \subset \deg^{-1}([1-\eta]) .$$

This fact has been proven in the talk 6.

Explain the abstract notion of a p-adic Banach space and what it means for an operator to be completely continuous, as in the beginning of Serre. Prove that U_p acts completely continuously on the space of overconvergent modular forms, as in Kisin-Lai, coro 4.3.6 (which is written for Hilbert modular varieties but translate for usual modular forms), giving all the results needed of the chapter 2 of Kisin-Lai.

Explain the canonical subgroup theory in the very rough following form which will be enough for us. First notice that an elliptic curve over $\mathrm{Spec}(\mathbb{Z}_p)$ with ordinary reduction has a unique subgroup in $E_{\mathbb{Q}_p}[p]$ with multiplicative reduction : this follows from the standard rigidity arguments for groups of multiplicative type (SGA 3). Therefore $\pi_1 : \bar{X}_0(p) \to \bar{X}$ which maps (E,H) to E induces an isomorphism between $\bar{X}_0(p)^{\mathrm{rig}}_{\mathrm{ord-mul}}$ and $\bar{X}^{\mathrm{rig}}_{\mathrm{ord}}$. This isomorphism extends to strict neighborhood by the proposition 2.2.4 of [PS2], proven in the talk 5. The inverse of this isomorphism associate to an elliptic curve E which is close to be ordinary a « canonical » subgroup of $E[p]$ which is close to be multiplicative. Remark : there exists complicated theories giving a numerical meaning to this closeness. This will not be important for us.

Corollary : overconvergent modular forms can also be seen as sections of ω^k on a strict neighborhood of $\bar{X}^{\mathrm{rig}}_{\mathrm{ord}}$ on $\bar{X}^{\mathrm{rig}}$. Therefore, it doesn't really makes sense to say that an overconvergent modular form has level $\Gamma_0(p)$ or no level at p.

Prove finally that ordinary p-adic modular forms in the sense of Hida (see talk 7) are overconvergent. Adapt [P2, th A.1.(3)] and its proof to usual modular forms. The equivalent of [P2, th.A.1.(1)(2)] for usual modular forms would have been already proven in talk 7. The proof of [P2, th A.1.(3)] should be very easy now and can be found on p.50 of this paper.

References : [K2], [S1], [KL], [P2].

5.2. **Talk 10 : Classicality in weight ≥ 2.** In this talk you will explain the proof of a famous theorem of Coleman saying that any overconvergent modular form f of weight $k \in \mathbb{Z}$ eigen for U_p is a classical modular form if the valuation of the eigenvalue α of U_p (which is called the slope) is $< k-1$.

You will not explain the original proof of Coleman but a proof found latter by Buzzard and Kassaei. This proof amounts to use the functional equation $U_p(f) = \alpha f$ to extends analytically f to $X_0(p)^{\mathrm{rig}}$. But this rigid variety is proper, we can apply GAGA and deduce the classicality of f.

First use results of the paragraph 2 of [P2], already stated and proven in talk 6. Apply [P2], prop 6.2 (more precisely, the translation of this statement for usual modular forms) to show that if $\alpha \neq 0$, the form f extends to $\deg^{-1}(]0,1])$.

It remains to extend f to $\deg^{-1}(0)$. This is done in Kassaei. First present the gluing lemma 2.3 of Kassaei (but do not spend to much time on 2.1, 2.2). Then explain Kassaei, paragraph 4. If you want to find another presentation, you can look at [P1] (which deals with Siegel modular forms but you can translate).

Remark that the theorem of Coleman is a motivation for studying overconvergent modular forms : in the proof one sees exactly where the overconvergence is important.

Remark also that for ordinary forms, the valuation of α is zero by definition. So Coleman's theorem apply for weights ≥ 2, but not to weight one. That's why Buzzard and Taylor had to found another way to prove classicality in weight one.

References : [K2], [P1].

5.3. **Talk 11 : Classicality in weight one.** Present Buzzard-Taylor work, more precisely [BT, th.4] and its proof, which is just one page long. Please, don't use the notations of Buzzard and Taylor, but the notations introduced in the previous talks, and especially the degree function of the talk 6.

The theorem of Buzzard and Taylor can be rephrased in the following way, with the same notations as in their theorem 4 : if f and g are two p-adic modular forms of level $\Gamma_0(p)$ which are ordinary for U_p such that $g = \pi_2^*(f)$, then both f and g are classical. Here π_1 and $\pi_2 : \bar{X}_0(p) \to \bar{X}$ are respectively the maps $(E, H) \mapsto E$ and $(E, H) \mapsto E/H$ outside the cusps. Moreover, as π_1 induces an isomorphism between the multiplicative ordinary locus $\bar{X}_0(p)^{\mathrm{rig}}_{\mathrm{ord-mult}}$ and the ordinary locus $\bar{X}^{\mathrm{rig}}_{\mathrm{ord}}$, we have seen f as a form (a section of ω^k where k is its weight) defined on $\bar{X}^{\mathrm{rig}}_{\mathrm{ord}}$.

The ordinarity condition for the action of U_p is the following, which is a little more general as what appeared in the previous talks (same difference as eigenvectors versus generalized eigenvectors). On says that a p-adic modular form g is ordinary for the Hecke operator U_p if there exists a polynomial $P \in \bar{\mathbb{Q}}_p[T]$ such that $P(U_p)(g) = 0$ and $P(0) \in \bar{\mathbb{Z}}_p^*$. By results of talk 9, this implies that g is overconvergent. In Buzzard-Taylor theorem, one has simply $P(T) = (T - \alpha)(T - \beta)$.

You can either follow Buzzard-Taylor proof, or find another proof which is an etale descent by π_2 (ask me for more details). Indeed because of results of the talk 6 and the ordinarity, g extends to $\deg^{-1}(]0,1])$. One will show that it still descends by π_2 on

$$\pi_2\left(\deg^{-1}(]0,1])\right) \; = \; \bar{X}^{\mathrm{rig}}$$

and therefore the classicity will follow by the GAGA principle. To show that g still descends by π_2 on $\deg^{-1}(]0,1])$ one can write the usual descent diagram for π_2

$$\begin{array}{ccccc}
\deg^{-1}(]0,1]) \times_{\pi_2,\, \bar{X}^{\mathrm{rig}},\, \pi_2} \deg^{-1}(]0,1]) & \Longrightarrow & \deg^{-1}(]0,1]) & \longrightarrow & \pi_2\left(\deg^{-1}(]0,1])\right) \; = \; \bar{X}^{\mathrm{rig}} \\
\uparrow & & \uparrow & & \uparrow \\
\bar{X}_0(p)^{\mathrm{rig}}_{\mathrm{ord-mult}} \times_{\pi_2,\, \bar{X}^{\mathrm{rig}},\, \pi_2} \bar{X}_0(p)^{\mathrm{rig}}_{\mathrm{ord-mult}} & \Longrightarrow & \bar{X}_0(p)^{\mathrm{rig}}_{\mathrm{ord-mult}} & \longrightarrow & \bar{X}^{\mathrm{rig}}_{\mathrm{ord}}
\end{array}$$

If one denotes by $p, q : \deg^{-1}(]0,1]) \times_{\pi_2,\, \bar{X}^{\mathrm{rig}},\, \pi_2} \deg^{-1}(]0,1]) \to \deg^{-1}(]0,1])$ one has by hypothesis $p^*(g) = q^*(g)$ on

$$\bar{X}_0(p)^{\mathrm{rig}}_{\mathrm{ord-mult}} \times_{\pi_2,\, \bar{X}^{\mathrm{rig}},\, \pi_2} \bar{X}_0(p)^{\mathrm{rig}}_{\mathrm{ord-mult}}$$

and it is enough to extends this equality on

$$\deg^{-1}(]0,1]) \times_{\pi_2,\, \bar{X}^{\mathrm{rig}},\, \pi_2} \deg^{-1}(]0,1])$$

By the analytic continuation principle, it is enough to prove that the canonical morphism from the set of connected components

$$\pi_0\left(\bar{X}_0(p)^{\mathrm{rig}}_{\mathrm{ord-mult}} \times_{\pi_2,\, \bar{X}^{\mathrm{rig}},\, \pi_2} \bar{X}_0(p)^{\mathrm{rig}}_{\mathrm{ord-mult}}\right) \to \pi_0\left(\deg^{-1}(]0,1]) \times_{\pi_2,\, \bar{X}^{\mathrm{rig}},\, \pi_2} \deg^{-1}(]0,1])\right)$$

is surjective. But this can be shown directly (try yo do it !)

6. Friday : The Artin conjecture

6.1. Talk 12 : Ordinary representations. This talk introduce general notions in p-adic Hodge theory ; this is more a survey than a talk with all the proofs provided.

Focus on a continuous Galois representation $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$. Recall the definition of being crystalline, semi-stable and potentially semi-stable [Fo, II and III]. What will be important for us is that we can construct a Weil-Deligne representation $D_{\mathrm{pst}}(\rho)$ which a $\bar{\mathbb{Q}}_p$-vector space of dimension ≤ 2, endowed with a nilpotent operator N and an action of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ which has finite image on the inertia. Then ρ is potentially semi-stable (or de Rham, this is equivalent by a celebrated theorem of Fontaine and Colmez) iff the dimension of $D_{\mathrm{pst}}(\rho)$ is two. Moreover it is cyistalline iff $N = 0$ and the discrete action of the local Galois group is unramified.

Explain that the p-adic etale cohomology of proper smooth varieties over $\mathrm{Spec}(\mathbb{Q}_p)$ with good reduction over $\mathrm{Spec}(\mathbb{Z}_p)$ is crystalline. If the variety has only semi-stable reduction then it is semi-stable. Both are difficult theorems due to Faltings, do not enter into the proof. Corollary : modular forms of weight ≥ 2 give rise to crystalline Galois representations (if the form has no level at p) or semi-stable representations (if the form has level $\Gamma_0(p)$). Moreover, weight one modular forms give rise to potentially semi-stable Galois representations, because they are known to be of finite image (see talk 4).

Introduce the concept of ordinary representations following [Fo, IV]. Give the main theorem of Perrin-Riou : ordinarity implies potentially semi-stable.

Introduce a generalisation of this concept : ρ is nearly ordinary if it is conjugated to an upper triangular representation of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. Note that this does not imply the potential semi-stability : on the opposite, the logarithm of the cyclotomic caracter can be used to produce an extension of the trivial character by itself which is not potentially semi-stable. Give the following theorem : if f is a p-adic modular form which is ordinary for the Hecke operator U_p (which means that $U_p(f) = \alpha \cdot f$ with $\alpha \in \bar{\mathbb{Z}}_p^*$) then the Galois representation ρ_f is nearly ordinary. But it is a priori not potentially semi-stable unless f is classical.

6.2. Talk 13 : Deformations rings. Stéphane Bijakowski

6.3. Talk 14 : R=T and conclusion. Benoît Stroh

Références

[B] P. Berthelot – *Cohomologie rigide et cohomologie rigide à support propre. Première partie*, preprint on perso.univ-rennes1.fr/pierre.berthelot/publis/

[BT] K. Buzzard, R. Taylor – *Companion forms and weight one forms*, Annals of Mathematics 149 (1999)

[C] C. Cogdell – *On the Artin L-functions*, on his webpage http ://www.math.osu.edu/ cogdell.1/artin-www.pdf

[DS1] F. Diamond, W. Shurman – *A first course in modular forms*, Springer.

[DS2] P. Deligne, J.P. Serre – *Formes modulaires de poids un*, Annales ENS.

[DR] P. Deligne, M. Rapoport – *Les schémas de modules des courbes elliptiques*, Modular functions in one variable II, Springer.

[Fa] L. Fargues – *La filtration de Harder-Narashiman des schémas en groupes finis et plats* , Crelle.

[Fo] J.M. Fontaine – *Périodes p-adiques*, Astérisque 223 (1994).

[KL] M. Kisin, Lai – *Overconvergent modular forms*, American Journal of Math.

[K1] N. Katz – *p-adic properties of modular schems*, Modular functions in one variable III, Springer.

[K2] P. Kassaei – *A gluing lemma and overconvergent modular forms*, Duke.

[N] J. Neukirch – *Algebraic number theory*, Springer.

[OT] F. Oort, J. Tate – , Annales ENS.

[P1] V. Pilloni – *Prolongement analytique sur les variétés de Siegel*, Duke.

[P2] V. Pilloni – *Sur la théorie de Hida*, Bulletin SMF.

[PS1] V. Pilloni, B. Stroh – *Surconvergence, ramification et modularité*, preprint.

[PS2] V. Pilloni, B. Stroh – *Surconvergence et classicité : le cas Hilbert*, preprint.

[S1] J. P. Serre – *Endomorphismes complètement continus des espaces de Banach p-adiques*, Publications IHES.

[S2] Takeshi Saito – *Galois representations and modular forms*, http ://www.ms.u-tokyo.ac.jp/ t-saito/talk/eepr.pdf

[SBT] N. Shepherd-Barron, R. Taylor – *Mod 2 and mod 5 icosahedral representations*, JAMS.

[T] R. Taylor – *On icosahedral Artin representations II*, American Journal of Math.