Special cycles on Shimura curves

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1 Introduction

This summer school is concerned with two results linking cycles on arithmetic models of orthogonal Shimura varieties to modular forms. The first result is [9, Theorem 3] while the second is [11, Theorem A]. We recommend reading the introductory chapters of [6], [9], [10] and [11] to get an overview of the involved statements and related results.

We now use this introduction to formulate both theorems and to outline the program for this school.

1.1 CM elliptic curves and incoherent Eisenstein series

Let $K/\mathbb{Q}$ be an imaginary-quadratic field with ring of integers $\mathcal{O}_K$. A *CM elliptic curve* over a scheme S over $\mathcal{O}_K$ is defined as follows. It is a pair (E, κ) where E/S is an elliptic curve and $\kappa : \mathcal{O}_K \to \mathrm{End}(E)$ is a ring homomorphism such that the induced action of $\mathcal{O}_K$ on $\mathrm{Lie}(E)$ is via the structure morphism. We denote by $M/\operatorname{Spec}\mathcal{O}_K$ the coarse moduli scheme of such pairs. By the main theorem of complex multiplication, $M \cong \operatorname{Spec}\mathcal{O}_H$ where H/K is the Hilbert class field.

A *special endomorphism* of a CM elliptic curve (E, κ) is an endomorphism $j \in \mathrm{End}(E)$ with $\mathrm{tr}(j) = 0$ and $j \circ \kappa(\alpha) = \kappa(\overline{\alpha}) \circ j$ for all $\alpha \in \mathcal{O}_K$. For each $t > 0$, we define the *special cycle* $Z(t) \longrightarrow M$ as the coarse moduli space of triples (E, κ, j) where (E, κ) is a CM elliptic curve with special endomorphism j of degree t. The scheme $Z(t)$ is a disjoint union of divisors on M. In particular, it is artinian and we define its arithmetic degree as

$$\widehat{\deg}(Z(t)) := \log|\mathcal{O}_{Z(t)}| = \sum_p \sum_{x \in Z(t)(\mathbb{F}_p^{\mathrm{alg}})} \mathrm{len}_{W(\mathbb{F}_p^{\mathrm{alg}})}(\widehat{\mathcal{O}}_{Z(t),x}) \cdot \log p.$$

Here, $\widehat{\mathcal{O}}_{Z(t),x}$ is the strict completion of $\mathcal{O}_{Z(t),x}$. The spectrum $\operatorname{Spec}\widehat{\mathcal{O}}_{Z(t),x}$ is the formal deformation space of the tripel $(E, \kappa, j)/\mathbb{F}_p^{\mathrm{alg}}$ defined by x.

For technical reasons, we assume from now on that the discriminant of K is prime. Then the first major result of this school is [6, Theorem 1.2.2] which gives the degree $\widehat{\deg}(Z(t))$. We will get this result by a direct computation as in [6]. First we count the geometric points $Z(t)(\mathbb{F}_p^{\mathrm{alg}})$ in Talk 2.2, then we compute the length of the local ring at

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each point in Talk 2.3. The calculation of the length reduces to the formula of Gross [5, Proposition 3.3] which we will prove by following the exposition in [1].

In the fourth talk 2.4, we will define a certain (non-holomorphic) modular form ϕ as in [9, §1 and §2]. This modular form is the central derivative of an incoherent Eisenstein series. A direct computation of its Fourier coefficients will prove our first main result:

Theorem 1.1 (Kudla-Rapoport-Yang [9]). *Let*

$$\phi(\tau) = \sum_{t \in \mathbb{Z}} a_t(\phi) q^t$$

be the Fourier expansion of ϕ, where $q = \exp(2\pi i \tau)$. Then for all $t > 0$,

$$a_t(\phi) = \widehat{\deg}(Z(t)).$$

1.2 Shimura curves and special cycles

Let $B/\mathbb{Q}$ be an indefinite quaternion division algebra with maximal order $\mathcal{O}_B^\times$. We denote by $\mathcal{M}/\operatorname{Spec}\mathbb{Z}$ the moduli stack of abelian surfaces with an action of $\mathcal{O}_B$ satisfying the *special condition*. It is a regular "surface" (two-dimensional stack), proper and flat over $\mathbb{Z}$ and smooth away from the primes dividing the discriminant $D(B)$ of B. In fact, it is an arithmetic model of the Shimura curve $\mathcal{O}_B^\times \backslash \mathfrak{h}^\pm$ associated to the quaternion algebra B.

The surface $\mathcal{M}$ has semi-stable reduction at the primes $p \mid D(B)$. Its fiber over such p is a configuration of lines, which intersect transversally. We will use the theorem of Cherednik-Drinfeld (i.e. p-adic uniformization by the formal Drinfeld upper half-plane $\widehat{\Omega}$) to study these fibers. This will also determine the subspace of $\mathrm{CH}^1(\mathcal{M})$ spanned by vertical cycles.

In Talk 3.3, we will define a family of divisors on $\widehat{\Omega}$ following [8]. These *local special cycles* are (up to Cohen-Macaulayfication) the loci on $\widehat{\Omega}$ where a given endomorphism of the framing object lifts. The local special cycles uniformize the global ones, which will be defined in the following talk.

In Talk 3.4, we will define the *special cycle* $\mathcal{Z}(t) \longrightarrow \mathcal{M}$ for each integer $t > 0$. As in the case of the moduli of CM elliptic curves, the support of $\mathcal{Z}(t)$ is the locus where the abelian surface has "complex multiplication" by the order $\mathbb{Z}[\sqrt{-t}]$. These cycles can have vertical components in the fibers of bad reduction. A careful analysis of these components with the help of the p-adic uniformization will prove the following result, see [11, Theorem 4.3.4].

Theorem 1.2. *Let Y be a vertical divisor on the surface $\mathcal{M}$ (i.e. a linear combination of components of closed fibers of $\mathcal{M}/\mathbb{Z}$). Denote by ω the relative dualizing bundle for $\mathcal{M}/\mathbb{Z}$ and denote by $\langle\ ,\ \rangle$ the usual intersection product of divisors on a surface. Then the series*

$$-\langle \omega, Y \rangle + \sum_{t > 0} \langle \mathcal{Z}(t), Y \rangle q^t \in \mathbb{C}[[q]] \tag{1.1}$$

is the q-expansion of an elliptic modular form of weight $3/2$.

Up to now, the theory was purely algebraic. But we will endow the $\mathcal{Z}(t)$ with Green functions on their complex fibers to define classes $\widehat{\mathcal{Z}}(t, v)$ in the arithmetic Chow group

$\widehat{\mathrm{CH}}^1(\mathcal{M})$. These classes are even defined for $t \leq 0$. We will recall the definition of $\widehat{\mathrm{CH}}^1(\mathcal{M})$ in Talk 4.1 and prove the important decomposition [11, Proposition 4.1.2]

$$\widehat{\mathrm{CH}}^1(\mathcal{M}) = \widetilde{\mathrm{MW}} \oplus (\mathbb{R}\widehat{\omega} \oplus \mathrm{Vert}) \oplus a(A^0(\mathcal{M}_{\mathbb{R}})_0) \tag{1.2}$$

in Talk 4.2. We can then form the generating series

$$\widehat{\phi}(q) := \sum_{t \in \mathbb{Z}} \widehat{\mathcal{Z}}(t, v) q^t \in \widehat{\mathrm{CH}}^1(\mathcal{M})[[q, q^{-1}]]. \tag{1.3}$$

The second main result of this school is then the following theorem.

Theorem 1.3 (Kudla-Rapoport-Yang [11]). *The generating series $\widehat{\phi}(q)$ is the Fourier expansion of a (non-holomorphic) modular form of weight $3/2$ and level $\Gamma_0(4D(B))$ with values in $\widehat{\mathrm{CH}}^1(\mathcal{M}) \otimes_{\mathbb{R}} \mathbb{C}$.*

Note that $\widehat{\phi}$ is a sum of four components corresponding to the decomposition (1.2). It is enough to prove the theorem for each component separately. The vertical component was already handled in previous talks, at least up to the claim about the level. The remaining talks are devoted to the proof of the modularity of the other components.

We will consider the Hodge component

$$\phi_{\mathrm{height}} := \langle \widehat{\phi}, \widehat{\omega} \rangle \in \mathbb{C}[[q, q^{-1}]]$$

in more detail. The proof of its modularity relies on an explicit computation of its Fourier coefficients. These can then be compared to the Fourier coefficients of the derivative of a certain Eisenstein series, see [10, Theorem A]. To calculate the coefficients of ϕ_{height}, one has to determine the Faltings heights of certain abelian surfaces which are isogeneous to the product of two CM elliptic curves. We will consider both the heights of CM elliptic curves and their change under isogeny.

The last two talks deal with the Mordell-Weil and the analytic component of $\widehat{\phi}$. We will follow the presentation given in [11]. Only one talk, if there is an introduction on Sunday.

1.3 Some remarks

All talks last 90 minutes. This program should be understood as a suggestion and speakers should feel free to change the emphasis of their talk. Please do not hesitate to contact us[1] if you have any questions.

2 Moduli of CM elliptic curves

2.1 Sunday: Moduli of CM elliptic curves and special cycles

Possibly an introductory talk, maybe by Kudla.

Follow the beginning of [9, §5] to define the moduli space $\mathcal{M}/\operatorname{Spec}\mathcal{O}_K$ of CM elliptic curves. State and prove Proposition 5.1. To show that $\mathcal{M}$ is étale over $\operatorname{Spec}\mathcal{O}_K$, proceed as follows.

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You need to show that the formal deformation space at a geometric point $(E, \kappa) \in \mathcal{M}(\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}})$ over $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$ is isomorphic to the strict completion $\widehat{\mathcal{O}}_{K,\mathfrak{p}}$. Recall the Theorem of Serre-Tate to reduce this to a statement about p-divisible groups. Briefly define the notion of a *formal $\mathcal{O}_K$-module of height h* as in [17, §1.1 and §1.2]. State [16, Theorem 3.8] without proof, then follow the case distinction in [9] after Corollary 5.2. In both cases, the connected part of the p-divisible group is a formal $\mathcal{O}_{K,\mathfrak{p}}$-module of height 1 and the cited theorem yields the result. The universal deformation of (E, κ) over $\widehat{\mathcal{O}}_{K,\mathfrak{p}}$ is called the *canonical lift*.

Define the space of special endomorphisms of a CM elliptic curve and define the special cycles $Z(m)$. State and briefly prove Proposition 1.1.3 and Corollary 1.1.4 in [6]. In particular if $m > 0$, then $Z(m)$ is a zero cycle. Define the arithmetic degree of $Z(m)$.

Introduce the set $\mathrm{Diff}(m)$ and state [6, Theorem 1.2.1]. Mention that this theorem yields an explicit formula for $\widehat{\deg}(Z(m))$ as in Theorem 1.2.2. Prove part a) of the theorem by following [6] through §1.4.

2.2 Monday I: Geometric points and Orbital integrals

The aim of this talk is to prove Theorem 1.2.1 b) of [6].

Define an action of $\mathrm{Pic}(\mathcal{O}_K)$ on $\mathcal{M}$ with the help of the Serre construction and recall the main theorem of complex multiplication as in Proposition 5.3 and Corollary 5.4 of [9] without proof. Prove Corollary 5.5 which is a crucial ingredient in the orbital integral formula. An argument for the proof is given in [6, Corollary 3.7.6], it uses the existence of canonical lifts from the first talk to reduce the statement to the generic fibre.

Follow [6, §2.2 and §2.3] to prove Theorem 1.2.1 b).

2.3 Monday II: Formula of Gross

The aim of this talk is to prove Theorem 1.2.1 c) of [6].

Use [9, Corollary 5.2] to explain why the length of $Z(m)$ at a geometric point $x \in Z(m)(\mathbb{F}_p^{\mathrm{alg}})$ can be computed as length of the deformation space of a triple (E, κ, j). (Note that $Z(m)$ is only a coarse moduli space.) Then use the theorem of Serre-Tate to reduce this to a statement about the canonical lift. To get the final formula [6, Theorem 3.10.1], you need to prove [17, Theorem 1.4] which is due to Gross [5]. You can proceed as follows.

Define formal group cohomology $H^2(F, M)$ as in [16]. Specialize to the case of height 2 and prove [17, Theorem 1.4] by induction. It is enough if you do this in the unramified case, the ramified case being similar. Note that the proof of [17, Lemma 1.3] is not valid since [16, Corollary 3.4] cannot be applied outside of characteristic p. We give an ad hoc proof in the appendix.

Formulate these results as Theorem 1.2.1 c) of [6] and give the formula of Theorem 1.2.2.

2.4 Monday III: Relation to Eisenstein series

The aim of this talk is to identify the length of $Z(m)$ with the m^{th} Fourier coefficient of the derivative of an incoherent Eisenstein series. This is [6, Proposition 1.3.4] which follows from Proposition 1.3.1.

Provide the general adelic setup from [9, §1]. Recall the Weil representation and define incoherent and coherent Eisenstein series. Specialize to our situation as in §2 and define the relevant Eisenstein series. Define its Fourier coefficients and explain how they factor in local Whittaker integrals.

Present the results of the computations of these Whittaker integrals (either from [6] or from [9, §2]) to deduce the equality $\deg(Z(m)) = -a'_m(v, 0)$.

3 Algebraic special cycles on Shimura curves

3.1 Tuesday I: Arithmetic models of Shimura curves

Define the Shimura curve $\mathcal{M}$ associated to an indefinite quaternion algebra $B/\mathbb{Q}$ as in [11, §3.1] and state Proposition 3.1.1. Sketch its proof, in particular explain the smoothness, the regularity and the properness with some more details than in [11]. For the properness, recall the valuative criterion for stacks, see [12]. Note that the Serre-Tate theorem and the deformation space of a p-divisible formal group of dimension 1 and height 2 [16, Theorem 3.8] were recalled in the first talk. The semi-stability will be proved in the next talk.

Explain the isomorphism of the quaternion Shimura datum with the orthogonal one as in [11, §3.2]. Explain the complex uniformization of $\mathcal{M}(\mathbb{C})$, i.e. Propositions 3.2.1 and 3.2.2.

Define the Hodge bundle on $\mathcal{M}$ and state that it is isomorphic to the relative dualizing bundle as in [10, §3]. Prove this at the primes of good reduction, where it follows from the functorial description of $\Omega^1_{\mathcal{M}/\mathbb{Z}}$.

3.2 Tuesday II: $\hat{\Omega}$ and p-adic uniformization

Briefly recall the notion of p-divisible group, isogeny and the rigidity theorem [13, Lemma 2.9]. Define special formal $\mathcal{O}_B$-modules and their moduli space $\mathcal{N}$ as in [8, §1].

Define the Bruhat-Tits tree $\mathcal{B}$ and the formal p-adic upper half-plane $\widehat{\Omega}$ as in [8]. Endow both $\mathcal{N}$ and $\widehat{\Omega}^\bullet$ with an action of $GL_2(\mathbb{Q}_p)$ and state the existence of a $GL_2(\mathbb{Q}_p)$-equivariant isomorphism between $\mathcal{N}$ and $\widehat{\Omega}^\bullet_W$. Describe this isomorphism on geometric points.

Explain the p-adic uniformization of $\mathcal{M}$ at a prime $p \mid D(B)$ as in [11, Theorem 3.2.3] and [4, Chapter III.5]. The crucial point here is the construction of the abelian surface with $\mathcal{O}_B$-action on $\widehat{\Omega}^\bullet_W \times H(\mathbb{A}_f^p)$. Take your time to explain this. Deduce that $\mathcal{M}$ has semi-stable reduction at the primes dividing $D(B)$.

3.3 Tuesday III: Special cycles on $\hat{\Omega}$

The reference for this talk is [8], the main results are Lemmas 4.9 and 6.2. They will be needed in the next talk to prove the modularity of various series.

Define the space V, give Definition 2.1, prove Proposition 2.1 and state Proposition 2.2. Your next aim is to determine the set $\mathcal{T}(j)$. Prove Lemma 2.7 and formulate Corollary 2.5.

Prove Theorem 3.1 which follows directly from the moduli description of $\widehat{\Omega}$. Use this to describe $Z(j)$ on the ordinary locus by equations. Omit the computation of $Z(j)$ at supersingular points. Give Proposition 3.2 and sketch its proof. Mention that there are embedded components at the superspecial points and that there can be an additional horizontal component. If you like, draw the picture from page 9.

Define the purification $Z(h)^{\text{pure}}$ and give Proposition 4.5. Recall the definition of the intersection number (4.6) for two closed formal subschemes Z, Z' such that the intersection $Z \cap Z'$ is a proper scheme over $\operatorname{Spf} W$. Note that this is additive for sums of formal divisors. You can also mention Lemma 4.3.

Conclude your talk by proving Lemma 4.9 and Lemma 6.2.

3.4 Wednesday: (Algebraic) Special cycles on $\mathcal{M}$ and their vertical components

Define the space of special endomorphisms of an abelian surface with action by $\mathcal{O}_B$ as in [11, §3.4]. Define the special cycle $\mathcal{Z}(t)$ and describe its complex uniformization. State the results about the degree of the horizontal component, see (3.4.4) and [10, Proposition 7.1]. Give [11, Prop 3.4.5] about the flatness of $\mathcal{Z}(t)$ away from $D(B)$.

Define the Cohen-Macaulayfication (or purification), see [10, page 54]. Explain the p-adic uniformization of $\mathcal{Z}(t)$ as in [11, §4.3]. State Proposition 4.3.2 and prove it for $p \neq 2$. Explain that the multiplicity $\mu_{[\Lambda]}(x)$ is induced from a Schwartz function on $V'(\mathbb{Q}_p)$. It is not so important for us to know this function explicitly.

Conclude your talk by stating and proving [11, Theorem 4.3.4]. We will show in later talks that ϕ_{Vert} is a component of $\widehat{\phi}$, just give (4.3.18) as an ad hoc definition.

4 Arithmetic cycles on Shimura curves

For basic definitions of Arakelov theory, see [3] or [15].

4.1 Thursday I: Intersection theory on regular surfaces

Recall the notion of a metrized line bundle on an arithmetic variety to define $\widehat{\text{Pic}}$. For a stack, the metric line bundle lives on a presentation and is endowed with a descent datum. As an important example, endow the Hodge bundle of an abelian scheme over a complex variety with a metric, see [11, §3.3].

Work through [11, §2.1] to define the Arakelov degree of metrized line bundles on one-dimensional Deligne-Mumford stacks. Mention the definitions from 2.2 (i.e. generalize the familiar notions from schemes to stacks.)

Define Green functions for divisors on orbifolds as in [11, §2.3]. Also recall the meaning of Green's equation.

Define the first arithmetic Chow group $\widehat{\text{CH}}^1_{\mathbb{Z}}(\mathcal{M})$ and give the isomorphism with $\widehat{\text{Pic}}(\mathcal{M})$. End your talk by defining the Arakelov height pairing as in [11, §2.6]. Mention that this defines the Arakelov intersection pairing on $\widehat{\text{CH}}^1$ by using the isomorphism with $\widehat{\text{Pic}}$, see (2.5.13).

4.2 Thursday II: Special cycles in arithmetic Chow groups

Endow the cycles $\mathcal{Z}(t)$ with Green functions and define their class in $\widehat{\mathrm{CH}}^1(\mathcal{M})$ as in [11, §3.5]. You can find more details on this in [7, §10 and §11]. Note that this also defines arithmetic cycles for $t < 0$. Give the definition of $\widehat{\mathcal{Z}}(0, v)$ from (4.2.4) and define the generating series $\widehat{\phi} = \widehat{\phi}_1$. State Theorem A, the central theorem of this summer school. Explain the meaning of this theorem, see the introduction of [11].

Prove the decomposition of $\widehat{\mathrm{CH}}^1(\mathcal{M})_{\mathbb{R}}$ as in [11, §4.1] with the desired amount of details. Make this decomposition explicit, see Proposition 4.1.4. Apply the decomposition to $\widehat{\phi}$ to get the component functions (4.2.7-11). Explain that the vertical component (4.2.9) is essentially the generating series from the talk of Wednesday, see (4.3.17-18).

4.3 Thursday III: Faltings heights of CM elliptic curves

The aim of this and the following talk is to compute the horizontal term in the Hodge component of $\widehat{\phi}$. Ideally, the speaker knows something about the Faltings heights of CM elliptic curves.

Start with the Hodge component of $\widehat{\phi}$ and deduce [10, (9.12)]. Recall the definition of the height of an abelian variety, see [10, (10.3)]. Follow the first chapter of Raynaud [14] to explain how the height changes under isogeny, see Proposition 1.4.1 there. You can restrict to the good reduction case. Also see [10, (10.18)].

Introduce the isogeny we are interested in from pages 43 and 44 of [10]. It involves two copies of an elliptic curve with CM by $\mathcal{O}_K$. Use the rest of the talk to say something about the computation of the height of a CM elliptic curve, see [10, Proposition 10.10].

4.4 Friday I: The change under isogeny

The aim of this talk is to compute the third summand in [10, (10.18)]. This will prove [10, Theorem 10.8] which explicitly computes one summand appearing in the Fourier coefficients of the Hodge component of $\widehat{\phi}$, see [10, (9.12)]. Note that the previous talk introduced most of §10 in [10] up to (10.19).

Reduce the computation of $\delta(u)$ to a computation with p-divisible groups. Recall the definition of the p-adic Tate module and its relation with isogenies as in §2 of [17]. You will also need Theorem 2.1 which describes isogenies with power series in a very explicit way.

Follow [10] in proving Theorem 10.8. Note that there are essentially three cases for $\delta_v(u)$. The most interesting one is Proposition 10.2 i), so you can concentrate on this case. For this, prove Proposition 10.3. Note that G_0 is the canonical lift, which is a Lubin-Tate module for $\mathcal{O}$. So for the Newton polygon argument, you can assume $[\pi](X) = X(\pi + X^{q-1})$, see [17].

Combine all results into Theorems 10.7 and 10.8. Note that $h^*_{\mathrm{Fal}}(E)$ was computed in the previous talk.

4.5 Friday II: Modularity of the Mordell-Weil component

Recall the definition of the Mordell-Weil component, see [11, (4.2.11)]. Follow Kudla through §4.5 and §4.6 of [11] to explain its modularity, see [11, Proposition 4.5.2]. If you want, you can also say something about the proof of Theorem 4.6.1. Once the existence of the Borcherds' lift is assumed, this proof is not very difficult, see [2].

4.6 Friday III: Modularity of the analytic component

Recall the definition of the analytic components, see (4.2.10) and (4.1.34)-(4.1.35) in [11]. Follow Kudla through 4.4 to prove the modularity of the analytic component with the desired amount of details. If there is an introduction on Sunday, this talk will be dropped.

5 Appendix

5.1 Proof for [17, Theorem 1.4] in the unramified case

Remark 5.1. In principle, we use the same line of reasoning as in §1 of [17]. But Lemma 1.3 there cannot be proved as stated since [16, Corollary 3.4.1] cannot be used outside of characteristic p. We implicitly give a correction here, at least for the unramified case.

Let L/K be an unramified extension of p-adic fields with uniformizer $\pi \in K$. Let $W/\mathcal{O}_L$ be the relative Witt vectors with maximal ideal $\mathfrak{m}$ and set $W_n := W/\mathfrak{m}^{n+1}$. Let F/W be a formal $\mathcal{O}_L$-module such that its reduction $F \otimes W_0$ is of height 1. In other words, F is a Lubin-Tate module for $\mathcal{O}_L$ and we can assume that multiplication by π is given by the power series

$$[\pi](X) := \pi X + X^{q^2}$$

where q is the cardinality of the residue field of K. We view F as the canonical lift for the underlying formal $\mathcal{O}_K$-module of height 2 with action of $\mathcal{O}_L$.

Let $F_n := F \otimes W_n$ and $H_n := \mathrm{End}(F_n)$. Reduction of endomorphisms defines injections (see Lemma 2.6 in [16]) $H_{n+1} \subset H_n$. It is known (see Theorem 1.1 of [17]) that H_0 is isomorphic to the maximal order in a quaternion division algebra over K. We choose a uniformizer $\Pi \in H_0$, i.e. an endomorphism of height 1.

Now let $f \in (\mathcal{O}_L + \Pi^l H_0) \setminus (\mathcal{O}_L + \Pi^{l+1} H_0)$. In other words, the height of f is odd and equals l.

Claim: $f \in H_{\lfloor l/2 \rfloor} \setminus H_{\lfloor l/2 \rfloor + 1}$ and the lift in $W_{\lfloor l/2 \rfloor}[[X]]$ has leading term αX^q with $v_\pi(\alpha) = \lfloor l/2 \rfloor$.

We first prove this for $l = 1$. Let

$$\tilde{f}(X) = \sum_{i=1}^{\infty} a_i X^i \in W[[X]]$$

be any (!) power series lift of f. Note that $v_\pi(a_i) \geq 1$ if $i \neq q$ and $v_\pi(a_q) = 0$. We compute

$$\begin{aligned} \delta_\pi(\tilde{f}) &= \tilde{f}(\pi X + X^{q^2}) -_F (\pi \tilde{f}(X) + \tilde{f}(X)^{q^2}) \mod \mathfrak{m}^2 \\ &\equiv -a_q \pi X^q \mod (\mathfrak{m}^2, X^{q+1}). \end{aligned}$$

So the cocycle never vanishes and thus $f \notin H_1$. The claim about the leading term is clear.

For the induction step, we need Lemma 3.1 [16]. Let f be of height $l+2 \geq 3$ and write $f = \pi g$ with g of height l. By induction, $g \in H_{\lfloor l/2 \rfloor} \setminus H_{\lfloor l/2 \rfloor + 1}$.

Let $\tilde{g} \in W[[X]]$ be a power series lift of $g \in H_{\lfloor l/2 \rfloor}$ and define

$$\varepsilon := \tilde{g} \circ [\pi] -_F [\pi] \circ \tilde{g}.$$

Since g is $\mathcal{O}_K$-linear, $\varepsilon \in \mathfrak{m}^{\lfloor l/2 \rfloor + 1}[[X]]$. Now $\tilde{f} := [\pi] \circ \tilde{g}$ is a power series lift of f and the power series $[\pi]$ commutes with the group law. (This is the crucial trick.) Then

$$\begin{aligned} \delta_\pi(\tilde{f}) &\equiv [\pi] \circ \varepsilon \mod \mathfrak{m}^{\lfloor l/2 \rfloor + 2} \\ &= 0. \end{aligned}$$

By Lemma 3.1, $f \in H_{\lfloor l/2 \rfloor + 1}$ and this lift is given by $[\pi] \circ \tilde{g}$, which also proves the claim about the leading term. To check that f does not lift further, one uses the same argument as in the case $l = 1$ using the the leading term.

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