Period differential equation for a family of $K3$ surfaces in connection with the Hilbert modular group of $\boldsymbol{Q}(\sqrt{5})$

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Outline

Algebraic $K3$ surface is characterized as a deformation of a nonsingular quartic surface in $\boldsymbol{P}^3$. It suggests that the $K3$ surface is a 2-dimensional analogy of the elliptic curve (namely the non singular cubic curve in $\boldsymbol{P}^2$).

The elliptic modular function $j(\tau)$ or $\lambda(\tau)$ is obtained as the inverse of the Schwarz map for a Gauss HGDE. In this case Gauss HGDE is a period differential equation for a family of elliptic curves.

On the other hand, a $K3$ surface S is characterized by the condition $K_S = 0$ and simply connected. It means that S is a two dimensional Calabi-Yau manifold. From this aspect the inverse Schwarz map is called "mirror map".

We are looking for a nice two dimensional analogy of this context. We say it the theory of "$K3$ modular function".

For this purpose we use the notion of 3-dimensional reflexive polytope with at most terminal singularity. Such a polytope P is defined by the intersection of several half spaces

$$a_i x + b_i y + c_i z \leq 1, (a_i, b_i, c_i) \in \boldsymbol{Z}^3, i = 1, \ldots, k$$

with the condition

(i) the origin is the unique inner lattice point,

(ii) only the vertices are the lattice points on the boundary.

We have 5 such polytopes with 5 vertices ([O]):

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}, P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix}, P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 \end{pmatrix},$$

$$P_5 = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}, P_r = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 \end{pmatrix},$$

here the column vectors indicate the vertices.

For every such a reflexive polytope we can find a corresponding 2-paramers family of $K3$ surfaces. We are going to study the family of $K3$ surfaces coming from P_4.

1 Family of $K3$ surfaces attached to P_4

As a principle we get the required family of algebraic surfaces by the following canonical procedure:

(i) Make a toric three fold X from the reflexive polytope. This is a rational variety with some singularity.

(ii) Take a divisor D on X that is linear equivalent to $-K_X$.

(iii) By the terminal condition D is represented by a $K3$ surface.

This procedure is realized by the following algebraic surfaces in $\boldsymbol{P}^3$ with 2 parameters λ, μ:

At first D is given by

$$a_1 + a_2 t_1 + a_3 t_2 + a_4 t_3 + a_5 \frac{1}{t_3} + a_6 \frac{1}{t_1 t_2 t_3^2} = 0$$

in (t_1, t_2, t_3)- space. Compare with

$$P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & -2 \end{pmatrix}.$$

By changing the variables and parameters at a sametime

$$\begin{cases} x = a_2 t_1 / a_1, \\ y = a_3 t_2 / a_1, \\ z = a_4 t_3 / a_1, \\ \lambda = a_4 a_5 / a_1^2, \\ \mu = a_2 a_3 a_4^2 a_6 / a_1^5 \end{cases} \tag{1.1}$$

we get a family

$$\mathscr{F} = \{S(\lambda, \mu)\}_{\lambda, \mu}$$

of $K3$ surfaces

$$S(\lambda, \mu) : xyz^2(x + y + z + 1) + \lambda xyz + \mu = 0. \tag{1.2}$$

We set

$$F(x, y, z) = xyz^2(x + y + z + 1) + \lambda xyz + \mu.$$

The projection map

$$S(\lambda, \mu) \to z\text{-plane}$$

make it an elliptic fibered surface over $\boldsymbol{P}^1$.

[About $K3$ surface S]

(a) There exists unique (up to constant factor) (non-vanishing) holomorphic 2-form ω.

(b) The 2nd homology group $H_2(S, \boldsymbol{Z})$ is a free $\boldsymbol{Z}$ module of rank 22.

(c) The intersection form of $H_2(S, \boldsymbol{Z})$ is isomorphic to $E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U$,

$$E_8(-1) = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}, \ U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(d) An elliptic fibered algebraic surface S over $\boldsymbol{P}^1$ is a $K3$ if and only if $\chi(S) = 24$, provided S is a minimal nonsingular model.

(e) Let $\{\Gamma_1, \dots, \Gamma_{22}\}$ be a basis of $H_2(S, \boldsymbol{Z})$.

$$\eta' = (\int_{\Gamma_1} \omega : \dots : \int_{\Gamma_{22}} \omega) \in \boldsymbol{P}^{21}$$

is said to be a periods of S.

(f) The Neron-Severi lattice $\mathrm{NS}(S)$ is the sublattice of $H_2(S, \boldsymbol{Z})$ generated by the divisors on S.

$\mathrm{T}(S) = \mathrm{NS}(S)^{\perp}$ is said to be the transcendental lattice of S. Let $\Gamma_1, \dots, \Gamma_r$ be a basis of $\mathrm{T}(S)$. Note

$$\int_{\Gamma} \omega = 0 \ \ \forall \Gamma \in \mathrm{NS}(S).$$

So the periods η' reduces to

$$\eta = (\int_{\Gamma_1} \omega : \dots : \int_{\Gamma_r} \omega) \in \boldsymbol{P}^{r-1}.$$

We note here that $\mathrm{NS}(S)$ is always with signature $(1, \cdot)$ and $\mathrm{T}(S)$ is always with signature $(2, \cdot)$.

(g) We have the Riemann-Hodge relation for the periods:

$$\eta' M^{-1t}\eta' = 0,$$
$$\eta' M^{-1t}\overline{\eta'} > 0,$$

where M is the intersection matrix $(\Gamma_i \cdot \Gamma_j)_{i,j}$. By this relation we always have the period in a type IV domain $\mathscr{D}$ of dimension $r-2$ dfined by

$$\mathscr{D} = \{\eta \in \boldsymbol{P}^{r-1} : \eta A^t \eta = 0, \eta A^t \overline{\eta} > 0\},$$

here A is the intersection matrix of $\mathrm{T}(S)$. Note that $\mathscr{D}$ is composed of two connected components $\mathscr{D}^+$ and $\mathscr{D}^-$.

2 Period map for $\mathscr{F}$

By some fiber preserving birational transformation, we can describe $S(\lambda, \mu)$ in the form

$$y_1^2 = 4x_1^3 - g_2(z)x_1 - g_3(z),$$

where $g_2(g_3, \text{resp.})$ is a polynomial of z of degree 8(12, resp.). This is an analogy of the Weierstrass normal form of the elliptic curve.

Set the discriminant $D_{x_1}(z) = g_2^3 + 27g_3^2$. We know that generically it has triple zero at $z = 0$, zero of multiplicity 15 at $z = \infty$ and other 6 simple zeros. So $S(\lambda, \mu)$ has singular fibres there. Sometimes we have a confluence of these singular fibers. We can observe them by the zero locus of the discriminant δ of $D_{x_1}(z)$ w.r.t. z.

Proposition 2.1. *We have the degenerating locus*

$$\delta = \lambda\mu(\lambda^2(4\lambda - 1)^3 - 2(2 + 25\lambda(20\lambda - 1))\mu - 3125\mu^2) = 0. \tag{2.1}$$

Outside δ, $S(\lambda, \mu)$ has a composition of singular fibers $I_3 + I_{15} + 6I_1$.

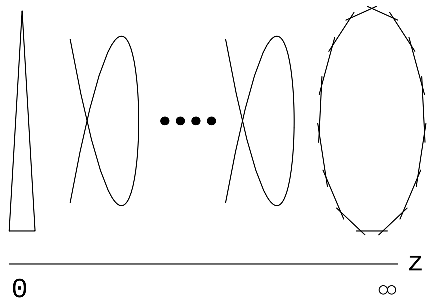

Figure 1. singular fibers of $S(\lambda, \mu)$

Consequently we have $\chi(S(\lambda, \mu)) = 24$, namely it is a $K3$ surface. The unique holomorphic form is given by

$$\omega = \frac{zdz \wedge dx}{\partial_y F}.$$

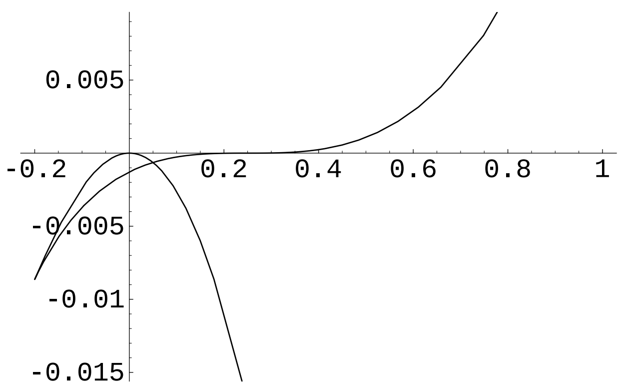

Figure 1. Degenerating locus δ

Set

$$\Lambda = \boldsymbol{C}^2 - \delta.$$

As a topological fiber space $\mathscr{F}_{|\Lambda} \to \Lambda$ is locally trivial. We have a parametrization

$$\lambda(a) = \frac{(a-1)(a+1)}{5},$$
$$\mu(a) = \frac{(2a-3)^3(a+1)^2}{3125}$$

of the third component of δ.

Theorem 2.1. *For a generic point λ, μ we have*

$$\mathrm{NS}(S(\lambda,\mu)) : M_0 \cong \begin{pmatrix} E_8(-1) & & & \\ & E_8(-1) & & \\ & & 2 & 1 \\ & & 1 & -2 \end{pmatrix}, \tag{2.2}$$

$$\mathrm{T}(S(\lambda,\mu)) : A \cong \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 2 & 1 \\ & & 1 & -2 \end{pmatrix}. \tag{2.3}$$

Definition 2.1. *Let $\Gamma_1, \ldots, \Gamma_4$ be a basis of $\mathrm{T}(S(\lambda,\mu))$ with*

$$(\Gamma_i \cdot \Gamma_j) = A.$$

We take periods $\eta(\lambda,\mu)$ in a neighborhood of one fixed point (λ,μ), and let us make the analytic continuation in Λ. Then we obtain a multivalued analytic map

$$\Phi : \Lambda \to \mathscr{D}.$$

We call it the period map for $\mathscr{F}$. Let us assume $\Phi(\Lambda) \subset \mathscr{D}^+$. By virtue of the Torelli theorem for $K3$ surfaces, we know that $\Phi(\Lambda)$ is dense open in $\mathscr{D}^+$.

3 Results

3.1 Period as a hypergeometric series

For a small parameters λ, μ, we have a lift of a small torus $\{|x| = \varepsilon\} \times \{|z| = \varepsilon\}$ ($\varepsilon > 0$) on $S(\lambda,\mu)$. It determines a 2-cycle $\Gamma \in H_2(S(\lambda,\mu), \boldsymbol{Z})$.

Theorem 3.1. *We have*

$$\eta(\lambda,\mu) = \int_\Gamma \omega = (2\pi i)^2 \sum_{n,m=0}^{\infty} (-1)^m \frac{(5m+2n)!}{n!(m!)^3(2m+n)!} \lambda^n \mu^m. \tag{3.1}$$

It is holomorphic in a neighborhood of the origin.

3.2 Period differential equation

Proposition 3.1. *Every period $\eta_1(\lambda,\mu), \dots, \eta_4(\lambda,\mu)$ satisfies the system of differential equations:*

$$\begin{cases} L_1 &= \theta_\lambda(\theta_\lambda + 2\theta_\mu) - \lambda(2\theta_\lambda + 5\theta_\mu + 1)(2\theta_\lambda + 5\theta_\mu + 2), \\ L_3 &= \lambda^2\theta_\mu^3 + \mu\theta_\lambda(\theta_\lambda - 1)(2\theta_\lambda + 5\theta_\mu + 1), \end{cases} \tag{3.2}$$

where $\theta_\lambda = \lambda\partial_\lambda, \theta_\mu = \mu\partial_\mu$.

This is a so called GKZ hypergeometric differential equation that is induced from the starting polytope in a usual way. But it has a 6-dimensional vector space of solutions. It is too much. Our vecor space generated by periods in a n.b.d. of a fixed point (λ,μ) is 4-dimensional. We can describe the required differential equation as its subsystem:

Theorem 3.2. (Period differential equation)

The system of periods $\{\eta_1(\lambda,\mu), \dots, \eta_4(\lambda,\mu)\}$ generates the the vector space of the solutions of the differential equation $L_1 = L_2 = 0$ with

$$\begin{cases} L_1 &= \theta_\lambda(\theta_\lambda + 2\theta_\mu) - \lambda(2\theta_\lambda + 5\theta_\mu + 1)(2\theta_\lambda + 5\theta_\mu + 2), \\ L_2 &= \lambda^2(4\theta_\lambda^2 - 2\theta_\lambda\theta_\mu + 5\theta_\mu^2) - 8\lambda^3(1 + 3\theta_\lambda + 5\theta_\mu + 2\theta_\lambda^2 + 5\theta_\lambda\theta_\mu) + 25\mu\theta_\lambda(\theta_\lambda - 1). \end{cases} \tag{3.3}$$

Consequently the projective monodromy group of Φ coincides with that of the system $L_1 = L_2 = 0$.

3.3 The Hilbert modular group as the monodromy group

Theorem 3.3. (Characterization of the monodromy group)

The projective monodromy group of Φ for the basis $\{\eta_1(\lambda,\mu), \dots, \eta_4(\lambda,\mu)\}$ is isomorphic to $PO(A, \boldsymbol{Z})^+ = \{g \in M(4, \boldsymbol{Z}) : {}^tgAg = A, g(\mathscr{D}^+) = \mathscr{D}^+\}/\pm I_4$.

Definition 3.1. *Let $\boldsymbol{H}^-$ be the lower half complex plane. Set $k = \boldsymbol{Q}(\sqrt{5})$, and set $\mathscr{O}_k = \boldsymbol{Z}[\frac{1+\sqrt{5}}{2}]$ be the ring of integers in k. The group $PSL(2, \mathscr{O}_k)$ acts on $\boldsymbol{H} \times \boldsymbol{H}^-$ by*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : (\tau_1, \tau_2) \mapsto \Big(\frac{\alpha\tau_1 + \beta}{\gamma\tau_1 + \delta}, \frac{\alpha'\tau_2 + \beta'}{\gamma'\tau_2 + \delta'}\Big) \tag{3.4}$$

here $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PSL(2, \mathscr{O}_k)$ and $'$ indicates the conjugate in k. In this situation we call $PSL(2, \mathscr{O}_k)$ the Hilbert modular group of k.

Theorem 3.4. *The pair $(\mathscr{D}^+, PO(A, \boldsymbol{Z})^+)$ is mudular equivalent over $\mathscr{O}_k$ with the Hilbert modular space $(\boldsymbol{H} \times \boldsymbol{H}^-, PSL(2, \mathscr{O}_k) \ltimes \langle\sigma\rangle)$, where*

$$\sigma : (\tau_1, \tau_2) \mapsto \left(\frac{1}{\tau_2}, \frac{1}{\tau_1}\right).$$

Remark 3.1. *We can describe the exact correspondence between above two moduli spaces as the following. Set*

$$W = \begin{pmatrix} (-1+\sqrt{5})/2 & (-1-\sqrt{5})/2 \\ 1 & 1 \end{pmatrix}.$$

It holds

$$A = U \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = U \oplus W(-U)^t W.$$

The correspondence

$$(\tau_1, \tau_2) \mapsto (I_2 \oplus {}^tW^{-1}) \begin{pmatrix} \tau_1\tau_2 \\ 1 \\ \tau_1 \\ \tau_2 \end{pmatrix}$$

defines a biholomorphic isomorphism

$$\iota : (\boldsymbol{H} \times \boldsymbol{H}^-) \cup (\boldsymbol{H}^- \times \boldsymbol{H}) \to \mathscr{D}.$$

Set $\rho(g) = \iota \circ g \circ \iota^{-1}$ for $g \in PSL(2, \mathscr{O}_k) \ltimes \langle\sigma\rangle$. In an explicit form we have

$$\rho\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) = (I_2 \oplus {}^tW)^{-1} \begin{pmatrix} \alpha & 0 & 0 & \beta \\ 0 & \delta & \gamma & 0 \\ 0 & \beta & \alpha & 0 \\ \gamma & 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} \alpha' & 0 & \beta' & 0 \\ 0 & \delta' & 0 & \gamma' \\ \gamma' & 0 & \delta' & 0 \\ 0 & \beta' & 0 & \alpha \end{pmatrix} (I_2 \oplus {}^tW).$$

and

$$\rho(\sigma) = H_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The pair (ι, ρ) gives the required isomorphism of the two modular spaces.

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