ON COMPOSITE RATIONAL FUNCTIONS HAVING A BOUNDED NUMBER OF ZEROS AND POLES

CLEMENS FUCHS AND ATTILA PETHŐ[‡]

Abstract. In this paper we study composite rational functions which have at most a given number of distinct zeros and poles. A complete algorithmic characterization of all such functions and decompositions is given. This can be seen as a multiplicative analog of a result due to Zannier on polynomials that are lacunary in the sense that they have a bounded number of non-constant terms.

1. Introduction and results

Let k be an algebraically closed field of characteristic zero and let $k(x)$ be the rational function field in one variable over k; for $f \in k(x)$ we define $\deg f = [k(x) : k(f(x))]$. We are interested in rational functions $f \in k(x)$ that are decomposable as rational functions, i.e. for which a relation of the form $f(x) = g(h(x))$ with $g, h \in k(x), \deg g, \deg h \geq 2$ holds. Observe that such a decomposition is only unique up to a linear fractional transformation $\lambda \in \mathrm{PGL}_2(k) = \mathrm{Aut}(\mathbb{P}^1(k))$ since we may always replace $g(x)$ by $g(\lambda(x))$ and $h(x)$ by $\lambda^{-1}(h(x))$ without affecting the equation $f(x) = g(h(x))$. Especially we are interested in such decompositions when f is a "lacunary" rational function.

There are different possible notions of "lacunarity". One way to define it, is to think of the number of terms appearing in a given representation of $f(x) = P(x)/Q(x), P, Q \in k[x]$ to be bounded. It was proved by Zannier in [8] that if one starts with a positive integer l, then one can describe effectively all decompositions of polynomials $f \in k[x]$ having at most l non-constant terms if one excludes the inner function h being of the exceptional shape $ax^n + b, a, b \in k, n \in \mathbb{N}$. Observe that for polynomials it is natural just to consider the non-constant terms since one can always replace f by $f - f(0)$ which has the same the degree as f and is composite if and only if f is. We also remark that it is enough to consider $g, h \in k[x]$ in this case and it is not hard to see that the exception is really needed. This description was "algorithmic" in the sense that all possible polynomials

Key words and phrases. Rational functions, decomposability, lacunarity, Siegel's identity, Mason-Stothers inequality, Brownawell-Masser inequality

2010 *Mathematics Subject Classification.* 11R58, 14H05, 12Y05.

[‡]Research of the second author was supported in part by the Hungarian National Foundation for Scientific Research grant No.T67580.

and decompositions were described by letting the possible coefficients vary in some effectively computable affine algebraic varieties and the exponents in some computable integer lattices. This gave a complete proof of a conjecture of Schinzel (saying that if for fixed g the polynomial $g(h(x))$ has at most l non-constant terms, then the number of terms of h is bounded only in terms of l) and more. The proof used as a first step an upper bound for the degree of g given only in terms of l that was already obtained by Zannier in [7]. (We remark that this last result was later generalized to Laurent-polynomials in [9] and then to rational functions in [3].) The further proof was a complicated inductive argument that used in its core an effective bound for the degrees of the solutions of S-unit equations (over function fields) in several variables due to Brownawell and Masser [1]; in fact a variant of it by Zannier [6] was used.

There are also other possibilities to think of the "lacunarity". In this paper we will be interested in rational functions f with a bounded number of zeros and poles (i.e. the number of distinct roots of P, Q in a reduced expression of f is bounded). As above we think of the number of zeros and poles as being fixed, whereas the actual values of the zeros and poles and their multiplicities are considered as variables. We also shall give a complete description of composite f's in analogy to Zannier's result in [6]; the proof of our result contains an algorithm that for given n provides all the data for rational functions f, g, h with f having at most n zeros and poles such that $f(x) = g(h(x))$ holds. We remark that a related type of question came up in [2] (see Proposition 2.4 therein).

Let us mention that we may assume (by changing $g(x)$ into $g(\theta x)$) that the rational function h is the quotient of two monic polynomials and by dividing both sides of the equation $f(x) = g(h(x))$ by a suitable constant we may even assume the same for f and g. (This is just to make the description below more readable). There are many trivial such families e.g. if the multiplicities of the zeros and poles of f all have a common divisor, say $m \in \mathbb{N}$, then $f(x) = (h(x))^m$ for some $h \in k(x)$; for this reason we say that if $g(x) = (\lambda(x))^m$ for a suitable $\lambda \in \mathrm{PGL}_2(k)$, then g is of exceptional shape (this has to be compared with the exceptional shape for h in the case above). We give a second example: Let λ_1, λ_2 be the roots of $x^2 - x - 1$ in $k = \mathbb{C}$ (so λ_1 is the golden mean), then for $g(x) = x^{k_1}(x-1)^{k_2}, h(x) = x(x-1)$ we have $f(x) = g(h(x)) = x^{k_1}(x-1)^{k_1}(x-\lambda_1)^{k_2}(x-\lambda_2)^{k_2}$ for every $k_1, k_2 \in \mathbb{Z}$. Thus we have constructed infinitely many rational functions f with four distinct zeros and poles altogether and which are decomposable.

The general situation is given in the following theorem, which is the main result of this paper:

Main Theorem. *Let n be a positive integer. Then there exists a positive integer J and, for every $i \in \{1, \dots, J\}$, an affine algebraic variety $\mathcal{V}_i$ defined over $\mathbb{Q}$ and with $\mathcal{V}_i \subset \mathbb{A}^{n+t_i}$ for some $2 \le t_i \le n$, such that:*

(i) *If $f, g, h \in k(x)$ with $f(x) = g(h(x))$ and with $\deg g, \deg h \ge 2$, g not of the shape $(\lambda(x))^m, m \in \mathbb{N}, \lambda \in \mathrm{PGL}_2(k)$, and f has at most n zeros and poles altogether, then there exists for some $i \in \{1, \dots, J\}$ a point $P = (\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_{t_i}) \in \mathcal{V}_i(k)$, a vector $(k_1, \dots, k_{t_i}) \in \mathbb{Z}^{t_i}$ with $k_1 + k_2 + \ldots + k_{t_i} = 0$ or not depending on $\mathcal{V}_i$, a partition of $\{1, \dots, n\}$ in $t_i + 1$ disjoint sets $S_\infty, S_{\beta_1}, \dots, S_{\beta_{t_i}}$ with $S_\infty = \emptyset$ if $k_1 + k_2 + \cdots + k_{t_i} = 0$, and a vector $(l_1, \dots, l_n) \in \{0, 1, \dots, n-1\}^n$, also both depending only on $\mathcal{V}_i$, such that*

$$f(x) = \prod_{j=1}^{t_i} (w_j/w_\infty)^{k_j}, \qquad g(x) = \prod_{j=1}^{t_i} (x - \beta_j)^{k_j},$$

and

$$h(x) = \begin{cases} \beta_j + \frac{w_j}{w_\infty} \ (j = 1, \dots, t_i), & \text{if } k_1 + k_2 + \cdots + k_{t_i} \neq 0, \\ \frac{\beta_{j_1} w_{j_2} - \beta_{j_2} w_{j_1}}{w_{j_2} - w_{j_1}} \ (1 \le j_1 < j_2 \le t_i), & \text{otherwise,} \end{cases}$$

where

$$w_j = \prod_{m \in S_{\beta_j}} (x - \alpha_m)^{l_m}, \quad j = 1, \dots, t_i$$

and

$$w_\infty = \prod_{m \in S_\infty} (x - \alpha_m)^{l_m}.$$

Moreover, we have $\deg h \le (n-1)/\max\{t_i - 2, 1\} \le n - 1$.

(ii) *Conversely for given data $P \in \mathcal{V}_i(k), (k_1, \dots, k_{t_i}), S_\infty, S_{\beta_1}, \dots, S_{\beta_{t_i}}, (l_1, \dots, l_n)$ as described in* (i) *one defines by the same equations rational functions f, g, h with f having at most n zeros and poles altogether for which $f(x) = g(h(x))$ holds.*

(iii) *The integer J and equations defining the varieties $\mathcal{V}_i$ are effectively computable only in terms of n.*

The example above is obtained by taking $n = 4, t = 2, S_\infty = \emptyset, S_0 = \{1, 2\}, S_\beta = \{3, 4\}, l_1 = l_2 = l_3 = l_4 = 1$ and $P = (0, 1, \lambda_1, \lambda_2, 1) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta) \in \mathcal{V}(\mathbb{C})$, where the variety $\mathcal{V} \subset \mathbb{A}^5$ is defined as the zero locus of the system of algebraic equations $\alpha_1\alpha_2 - \alpha_3\alpha_4 - \beta = 0, \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = 0$.

We mention that for $g(x) = g_1(x)/g_2(x), g_1, g_2 \in k[x]$ coprime and $\deg g_1 = \deg g_2$, then every pole of h will be cancelled in the decomposition $f(x) = g(h(x))$ and so a priori h could have arbitrary poles; this explains the difference between the two cases. We also mention that if additionally the number of distinct zeros and poles of g is two, then g has exactly one zero and one pole both with the same multiplicity and then we are in the forbidden shape for g.

The theorem can be seen as a multiplicative analog compared to the question studied by Zannier in the case that the number of non-constant terms is fixed. This already suggests that the proof is much easier. The essential part of it is to show that the inner function h has its degree bounded only in terms of n; for this we use a reduction to a system of two-dimensional S-unit equations, now over the rational function field, to which variants of the Brownawell-Masser inequality can be applied.

Before we state these crucial results (namely Zannier's variant from [6] and the Mason and Stothers theorem [4]), we briefly recall the theory of valuations on $k(x)$ (see [5]). For every $\theta \in k$ there is a valuation defined by the order of vanishing of f at $x = \theta$; moreover, for $f(x) = P(x)/Q(x), P, Q \in k[x]$ a non-archimedean valuation is defined by $v_\infty(f) = \deg Q - \deg P$. In this way all valuations $\mathcal{M}$ of $k(x)$ are obtained. Then we have

$$\deg f = \sum_{v \in \mathcal{M}} \max\{0, v(f)\} = -\sum_{v \in \mathcal{M}} \min\{0, v(f)\};$$

in other words the degree is just the number of zeros respectively poles of f (in $\mathbb{P}^1(k)$) counted by their multiplicities. The Mason-Stothers theorem now says that *for every $f, g \in k(x)$, not both constant, we have $\max\{\deg f, \deg g\} \leq |S| - 2$, where S is any set of valuations of $k(x)$ containing all the zeros and poles in $\mathbb{P}^1(k)$ of f and g.* (We remark that the upper bound is best possible). More generally by Zannier's variant of the Brownawell-Masser inequality, *if $g_1, \ldots, g_m \in k(x)$ span a k-vector space of dimension $\mu < m$ and any μ of the g_i are linearly independent over k, then*

$$-\sum_{v \in \mathcal{M}} \min\{v(g_1), \ldots, v(g_m)\} \leq \frac{1}{m - \mu} \binom{\mu}{2} (|S| - 2),$$

where S is any set of valuations of $k(x)$ containing all the zeros and poles in $\mathbb{P}^1(k)$ of $g_1, \ldots, g_m$. (This is [6, Theorem 2].) Now we are ready to give the proof of the theorem; this will be done in the next section.

2. Proof of the Main Theorem

Let n be a positive integer. We start with (i), so let $f, g, h \in k(x), \deg g, \deg h \geq 2, g$ not of the exceptional shape $(\lambda(x))^m, m \in \mathbb{N}, \lambda \in \mathrm{PGL}_2(k)$ and with f having at most n zeros and poles in $\mathbb{A}^1(k)$ altogether and such that $f(x) = g(h(x))$. Since k is algebraically closed we can write

$$f(x) = \prod_{i=1}^{n} (x - \alpha_i)^{f_i}$$

with pairwise distinct $\alpha_i \in k$ and $f_i \in \mathbb{Z}$ for $i = 1, \ldots, n$. (Remember that without loss of generality we are assuming that f is monic.) Similarly we get

$$g(x)=\prod_{j=1}^{t}(x-\beta_j)^{k_j} \tag{1}$$

with pairwise distinct $\beta_j \in k$ and $k_j \in \mathbb{Z}$ for $j = 1, \ldots, t$ and $t \in \mathbb{N}$. Thus we have

$$\prod_{i=1}^{n}(x-\alpha_i)^{f_i} = f(x) = g(h(x)) = \prod_{j=1}^{t}(h(x)-\beta_j)^{k_j}.$$

We now distinguish two cases depending on $k_1 + k_2 + \cdots + k_t \neq 0$ or not; observe that this condition is equivalent to $v_\infty(g) \neq 0$ or not. We shall write $h(x) = p(x)/q(x)$ with $p, q \in k[x], p, q$ coprime.

First assume that $v_\infty(g) \neq 0$. It follows that the poles in $\mathbb{A}^1(k)$ of h are among the values α_i: This is true because $q(\theta) = 0$ for $\theta \in k$ implies $h(\theta) = \infty$, where $\infty = (0 : 1)$ is the unique point at infinity of $\mathbb{P}^1(k)$, and $h(\theta) - \beta_j = \infty$. Also the valuation v_θ of h and $h(x) - \beta_j$ is the same. Thus $v_\theta(f) = v_\infty(g)v_\theta(h) \neq 0$, i.e. $g(h(\theta)) \in \{0, \infty\}$, and hence $\theta = \alpha_i$ for some $i \in \{1, \ldots, n\}$. This implies that there is a subset S_∞ of the set $\{1, \ldots, n\}$ such that the α_m for $m \in S_\infty$ are precisely the poles in $\mathbb{A}^1(k)$ of h, i.e.

$$q(x) = \prod_{m\in S_\infty}(x-\alpha_m)^{l_m}$$

for some $l_m \in \mathbb{N}$. Furthermore h and every function $h(x)-\beta_i$ have exactly the same poles in $\mathbb{P}^1(k)$ and again at a pole the multiplicities are equal; especially this implies that h and $h(x) - \beta_j$ have the same number of poles counted by multiplicity, which means that their degrees are equal. Calculating the valuations v_{α_m} of both sides of the equation $f(x) = g(h(x))$ we infer that $(k_1 + k_2 + \cdots + k_t)l_m = v_\infty(g)v_{\alpha_m}(h) = v_{\alpha_m}(f) = f_m$ for $m \in S_\infty$. We also point out that for $\beta_i \neq \beta_j$ the factors $h(x)-\beta_i$ and $h(x)-\beta_j$ do not have any zeros (in $\mathbb{A}^1(k)$) in common; therefore we have $t \leq n$. Now it follows that there is a partition of the set $\{1, \ldots, n\}\backslash S_\infty$ in t disjoint subsets $S_{\beta_1}, \ldots, S_{\beta_t}$ such that

$$h(x) = \beta_j + \frac{1}{q(x)}\prod_{m\in S_{\beta_j}}(x-\alpha_m)^{l_m}, \tag{2}$$

where $l_m \in \mathbb{N}$ satisfies $l_m k_j = f_m$ for $m \in S_{\beta_j}$, and this holds true for every $j = 1, \ldots, t$. Since we assume that g is not of the shape $(\lambda(x))^m$ it follows that $t \geq 2$. Let $1 \leq i < j \leq t$ be given. We have at least two different representations of h and thus we get

$$\beta_i + \frac{1}{q(x)}\prod_{r\in S_{\beta_i}}(x-\alpha_r)^{l_r} = \beta_j + \frac{1}{q(x)}\prod_{s\in S_{\beta_j}}(x-\alpha_s)^{l_s}$$

or equivalently $\beta(u_i - u_j) = 1$, where $\beta = 1/(\beta_j - \beta_i)$ and

$$u_i = \frac{1}{q(x)} \prod_{r \in S_{\beta_i}} (x - \alpha_r)^{l_r} = \frac{w_i}{w_\infty},$$

where for the last equality we have used the definition from the theorem; this is a non-constant rational function in $k(x)$ (otherwise h would be constant, but $\deg h \geq 2$ by assumption). Actually, the u_i are S-units (and the same is true for f) for the set of valuations $S = \{v_{\alpha_1}, \ldots, v_{\alpha_n}, v_\infty\} \subset \mathcal{M}$ corresponding to $\alpha_1, \ldots, \alpha_n \in k$ and ∞ (recall that $u \in k(x)$ is called an S-unit if $v(u) = 0$ for all $v \notin S$). In fact u_i and u_j have also no zeros in $\mathbb{A}^1(k)$ in common and they have all exactly the same poles (also with multiplicities), namely $\alpha_m, m \in S_\infty$ and possibly ∞. The Mason-Stothers theorem implies that

$$l_m \leq n - 1 \text{ for all } m = 1, \ldots, n. \tag{3}$$

Observe that an application to $\beta(u_i - u_j) = 1$ gives the bound only for those m which are in $S_\infty \cup S_{\beta_i} \cup S_{\beta_j}$; by using the relations from (2) for all possible combinations of $1 \leq i < j \leq t$ we see that indeed (3) holds. More precisely, it follows that L^+, the sum over all $l_m, m \in S_{\beta_i}$ plus $\max\{0, v_\infty(u_i)\}$, and L^-, the sum over all $l_m, m \in S_\infty$ plus $-\min\{0, v_\infty(u_i)\}$, is bounded by $n - 1$. This can be immediately improved by an application of [6, Theorem 2]: First let us define $u_{t+1} := 1$. The k-vector space generated by the S-units $u_1, \ldots, u_t, u_{t+1} \in k(x)$ has dimension 2 and any two of the u_i are linearly independent, because $\alpha u_i + \beta u_j = 0$ with $\alpha, \beta \in k$ implies either $u_i \in k$, a contradiction, or $\alpha u_i + \beta(u_i - \beta_j + \beta_i) = (\alpha + \beta)u_i + \beta(\beta_i - \beta_j) = 0$ and thus $\alpha = \beta = 0$. It follows that

$$\deg u_i = L^+ = L^- \leq - \sum_{v \in \mathcal{M}} \min\{v(u_1), \ldots, v(u_t), 0\} \leq \frac{n-1}{t-1} \leq n - 1$$

for all $i = 1, \ldots, t$. Especially, the degree of h is therefore also bounded by $n - 1$ since it is equal to the degree of u_i for all $i = 1, \ldots, t$, so altogether $\deg h = \deg u_i \leq (n-1)/(t-1) \leq n - 1$. By comparing coefficients in (2) after cancelling denominators for all combinations of the equations that have to hold there, we get an affine algebraic variety $\mathcal{V}$ (possibly reducible) defined over $\mathbb{Q}$ in the variables $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_t$; thus $\mathcal{V} \subset \mathbb{A}^{n+t}$. We point out that the number of variables and the exponents depend only on n. Since $f(x) = g(h(x))$ is given at this point, there are k-rational points on this algebraic variety and one of them corresponds to $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_t)$ coming from f and g.

Now we turn to the case $v_\infty(g) = 0$. Here we have

$$\prod_{i=1}^{n} (x - \alpha_i)^{f_i} = \prod_{j=1}^{t} \left(\frac{p(x)}{q(x)} - \beta_j \right)^{k_j} = \prod_{j=1}^{t} (p(x) - \beta_j q(x))^{k_j}.$$

Observe that a priori we have no control on the poles of h. However, as the factors on the right hand side of the last equation are again pairwise coprime, there is a partition of the set $\{1, \dots, n\}$ in t disjoint subsets $S_{\beta_1}, \dots, S_{\beta_t}$ such that

$$(p(x) - \beta_j q(x))^{k_j} = \prod_{m \in S_{\beta_j}} (x - \alpha_m)^{f_m}.$$

Thus k_j divides f_m for all $m \in S_{\beta_j}$ and this is true for all $j = 1, \dots, t$. On putting $l_m = f_m/k_j$ for $m \in S_{\beta_j}$ we obtain

$$p(x) - \beta_j q(x) = \prod_{m \in S_{\beta_j}} (x - \alpha_m)^{l_m} \tag{4}$$

for $j = 1, \dots, t$. Note that the exponents $l_m \in \mathbb{N}$, because $p(x) - \beta_j q(x)$ are polynomials and the α_m's are distinct. We have already pointed out above that in this case we may assume that $t \geq 3$, since g is not of exceptional shape. Let us choose $1 \leq j_1 < j_2 < j_3 \leq t$. From the corresponding three equations in (4) the so called Siegel identity $v_{j_1,j_2,j_3} + v_{j_3,j_1,j_2} + v_{j_2,j_3,j_1} = 0$ follows, where

$$v_{j_1,j_2,j_3} = (\beta_{j_1} - \beta_{j_2}) \prod_{m \in S_{\beta_{j_3}}} (x - \alpha_m)^{l_m}.$$

The quantities v_{j_1,j_2,j_3} are non-constant rational functions and they are S-units. Observe that by taking $j_1 = 1, j_2 = i, j_4 = j$ with $1 \leq i < j \leq t$ the Siegel identity can be rewritten as

$$\frac{\beta_j - \beta_1}{\beta_j - \beta_i} \frac{w_i}{w_1} + \frac{\beta_1 - \beta_i}{\beta_j - \beta_i} \frac{w_j}{w_1} = 1,$$

where we are using the definition of the w_i from the theorem. Moreover, we get from (4) that

$$\begin{aligned} p(x) &= \frac{1}{\beta_i - \beta_j} \left(\beta_i \prod_{m \in S_{\beta_j}} (x - \alpha_m)^{l_m} - \beta_j \prod_{m \in S_{\beta_i}} (x - \alpha_m)^{l_m} \right) \\ &= \frac{\beta_i w_j - \beta_j w_i}{\beta_i - \beta_j} \end{aligned} \tag{5}$$

and

$$q(x) = \frac{1}{\beta_i - \beta_j} \left(\prod_{m \in S_{\beta_j}} (x - \alpha_m)^{l_m} - \prod_{m \in S_{\beta_i}} (x - \alpha_m)^{l_m} \right) = \frac{w_j - w_i}{\beta_i - \beta_j}. \tag{6}$$

Hence, the numerator of h is in any case given by f, g and the integer vector $(l_1, \dots, l_n)$. The Mason-Stothers theorem applied to the Siegel identity now implies that $l_m \leq n - 1$ for $m \in S_{\beta_1} \cup S_{\beta_i} \cup S_{\beta_j}$; as we may choose e.g. $i = 2$ and $j = 3, \dots, t$ we have actually $l_m \leq n - 1$ for $m \in \{1, \dots, n\}$. More precisely it follows for every i that the sum over all l_m with $m \in S_{\beta_i}$ is bounded by $n-1$, hence by (5) and (6) it follows that the degrees of p, q and hence, since the degree of a rational function is the maximum of the degrees of the numerator and denominator in a reduced representation, the degree of h is bounded by $n-1$. Again this can be improved: We take $w_{t+1} := w_1$. Then the S-units $w_2/w_1, \ldots, w_t/w_1, w_{t+1}/w_1 = 1$ span a k-vector space of dimension 2 and any two are linearly independent, because the w_i are pairwise coprime polynomials and a constant quotient w_i/w_1 would imply that h is constant, a contradiction, and if $\alpha w_i/w_1 + \beta w_j/w_1 = 0$ for $1 \leq i < j \leq t$, then $(\alpha - \beta(\beta_j - \beta_1)/(\beta_1 - \beta_i))w_i/w_1 + \beta(\beta_j - \beta_i)/(\beta_1 - \beta_i) = 0$ and therefore $\beta(\beta_j - \beta_i) = 0$ which implies $\beta = \alpha = 0$. Thus [6, Theorem 2] gives that $\deg w_i/w_1 \leq (n-1)/(t-2)$ and, again since the w_i are coprime polynomials, $\deg w_i \leq (n-1)/(t-2)$ for all $i = 1, \ldots, t$. The definition of h now implies that $\deg h \leq (n-1)/(t-2) \leq n-1$. By taking the Siegel identities as defining equations we again get an algebraic variety $\mathcal{V} \subset \mathbb{A}^{n+t}$ and $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_t)$ is a k-rational point on this variety.

Finally we point out that we have $h(x) = \beta_j + w_j/w_\infty$ if $v_\infty(g) \neq 0$ and $S_\infty = \emptyset$ and $h(x) = (\beta_i w_j - \beta_j w_i)/(w_j - w_i)$ otherwise. In conclusion we have now proved (i).

Now we come to (ii) and (iii). The point is that we get all possible decompositions of rational functions with at most n zeros and poles altogether by considering for every integer $2 \leq t \leq n$ and for every partition of $\{1, \ldots, n\}$ into $t+1$ disjoint sets $S_\infty, S_{\beta_1}, \ldots, S_{\beta_t}$ (observe that their number is bounded only in terms of n) and for every choice of $(l_1, \ldots, l_n) \in \{0, 1, \ldots, n-1\}^n$ (here we use the crucial bound obtained in both cases; see e.g. (3)) the variety defined by equating the coefficients given by (2) after cancelling denominators and, if $S_\infty = \emptyset$ and $t \geq 3$, the variety given by the various Siegel identities. If the first system has a k-rational solution, then (2) defines the rational function $h(x)$; afterwards for any choice of integers $k_1, \ldots, k_t$ with $k_1 + \ldots + k_t \neq 0$ we define a rational function $g(x)$ by (1). If the second system has a k-rational solution, then we define $h(x) = p(x)/q(x)$ by (5) and (6) and then for any choice of integers $k_1, \ldots, k_t$ with $k_1 + \cdots + k_t = 0$ we define a rational function $g(x)$ again by (1). Finally, in both cases, we use

$$f(x) = \prod_{j=1}^{t} \left(\prod_{m \in S_{\beta_j}} (x - \alpha_m)^{l_m} \prod_{m \in S_\infty} (x - \alpha_m)^{-l_m} \right)^{k_j} = \prod_{j=1}^{t} (w_j/w_\infty)^{k_j}$$

to define the rational function f, which then has at most n zeros and poles altogether and for which $f(x) = g(h(x))$ holds. The number J of possible varieties is at most $2np(n)n^n$, where $p(n)$ is the partition function and since everything above is completely explicit, the defining equations of the varieties can be found explicitly. This proves the remaining parts of the statement. □

3. Acknowledgments

This paper was written during a visit of the first author at the University of Debrecen in November 2009; he is most grateful, especially to Ákos

Pintér, for the invitation and the hospitality during the stay. The authors are grateful to Umberto Zannier for valuable comments on an earlier version of the paper.

References

[1] D. Brownawell, D. Masser, Vanishing sums in function fields, *Math. Proc. Camb. Phil. Soc.* **100** (1986), 427-434.
[2] C. Fuchs, A. Pethő, R.F. Tichy, On the Diophantine equation $G_n(x) = G_m(P(x))$: higher-order recurrences, *Trans. Amer. Math. Soc.* **355** (2003), no. 11, 4657-4681.
[3] C. Fuchs, U. Zannier, Composite rational functions expressible with few terms, Preprint, 2009.
[4] R.C. Mason, *Diophantine equations over function fields*, London Mathematical Society Lecture Note Series **96**, Cambridge University Press, Cambridge, 1984.
[5] H. Stichtenoth, *Algebraic Function Fields and Codes*, Springer-Verlag, Universitext, 1993.
[6] U. Zannier, Some remarks on the S-unit equation in function fields, *Acta Arith.* **LXIV** (1993), no. 1, 87-98.
[7] U. Zannier, On the number of terms of a composite polynomial, *Acta Arith.* **127** (2007), no. 2, 157-167.
[8] U. Zannier, On composite lacunary polynomials and the proof of a conjecture of Schinzel, *Invent. Math.* **174** (2008), no. 1, 127-138.
[9] U. Zannier, Addendum to the paper: On the number of terms of a composite polynomial, *Acta Arith.* **141** (2009), no. 1, 93-99.

Clemens Fuchs
Department of Mathematics
ETH Zurich
Rämistrasse 101
8092 Zürich
Switzerland
Email: clemens.fuchs@math.ethz.ch

Attila Pethő
Department of Computer Science
University of Debrecen
Number Theory Research Group, Hungarian Academy of Sciences and University of Debrecen
P.O. Box 12
4010 Debrecen
Hungary
Email: petho.attila@inf.unideb.hu