ANGLE TRISECTION WITH ORIGAMI AND RELATED TOPICS

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Abstract. It is well known that the trisection of an angle with compass and ruler is not possible in general. What is not so well known (even if it is folklore in the community of geometric constructions and mathematical paper folding) is that angle trisection can be done with other tools, especially by an Origami construction. In this paper we briefly discuss this construction and give a geometric and an algebraic proof that the construction is correct. We also discuss which kind of problems Origami can solve and which other tools can be used to trisect an angle.

2000 *Mathematics Subject Classification:* 51M15, 11R04, 11R32.
Keywords: Angle trisection, ruler and compass, mathematical paper folding, Origami geometry, Galois theory.

1. Introduction

One of the famous old problems from antiquity is to trisect a given angle by using only a compass and a ruler. In other words: By using only a tool to draw a straight line segment through any two points and a tool to draw circles and arcs and duplicating lengths, one wants to construct in a finite number of steps two half-lines that comprise one third of an angle that itself is already given in terms of two half lines comprising it. The proof that this venture is indeed impossible in general is a prime example of how axiomatic mathematics works.

We start with algebraization of this problem. Let a subset $\mathcal{M} \subseteq \mathbb{A}^2(\mathbb{R})$ of the affine plane and two different points $0, 1 \in \mathcal{M}$ be given. The fact that we have 0 and 1 is equivalent to the existence of a coordinate system including the unity length by setting $(0,0) = 0, (1,0) = 1$. A line in the affine plane is given by two points. A circle is given by the mid-point and two other points the distance between them being the radius of the circle. In the coordinate system the points on a line satisfy a linear equation and the points on a circle can be described as the solution set of a quadratic equation. Let $\overline{\mathcal{M}}$ be the set of all points that can be constructed in a finite number of steps starting with points from $\mathcal{M}$ by the following axioms:

[†]This paper was initiated by a TV report "Origami löst unlösbare Probleme" in the science magazine *Einstein* of the Swiss national television SF1 broadcasted on April 9, 2009, in which the author got involved after a request of the producer.

(1) For any two lines each given by two points from $\overline{\mathcal{M}}$, we can construct the intersection point.
(2) Given a circle and a line both defined by points from $\overline{\mathcal{M}}$, then we can construct all intersection points of the circle with the line.
(3) Given two non-identical circles by points from $\overline{\mathcal{M}}$, then we can construct all the intersection points of the two circles.

In fact it is easy to see that (3) follows from (1) and (2) and that, by the Mohr-Mascheroni theorem (cf. [23, 16]), all constructions can be formulated by using the compass alone. An element $z \in \mathbb{R}$ is called constructable if the point $(z, 0)$ is in $\overline{\mathcal{M}}$. Let $\mathcal{K}(\mathcal{M})$ denote the set of all constructable elements and $\mathbb{Q}(\mathcal{M})$ the field generated over $\mathbb{Q}$ by the coordinates of the points in $\mathcal{M}$. It is well known that the following statements are equivalent

(i) $z \in \mathcal{K}(\mathcal{M})$
(ii) There is a sequence of quadratic field extensions $K_0 := \mathbb{Q}(\mathcal{M}) \subset K_1 \subset K_2 \subset \cdots \subset K_n \subset \mathbb{R}$ with $z \in K_n$.
(iii) z is contained in a Galois-extension K of $\mathbb{Q}(\mathcal{M})$ with $[K : \mathbb{Q}(\mathcal{M})]$ being a power of 2.

The equivalence of (i) and (ii) is elementary; the equivalence to (iii) however involves quite some algebra, especially Galois theory and the fact that every 2-group is solvable.

By using the Chebyshev identity $\cos(3\varphi) = 4\cos^3(\varphi) - 3\cos(\varphi)$, we conclude from the equivalences above that the angle $\theta = 3\varphi$ can be trisected by compass and ruler in a finite number of steps if and only if the polynomial $4X^3 - 3X - \cos(\theta)$ is reducible over the field $\mathbb{Q}(\cos(\theta))$. Especially, we get that $\frac{\pi}{3}$ cannot be constructed by compass and ruler since $X^3 - 3X - 1$ and so also $4X^3 - 3X - \frac{1}{2}$ is irreducible over $\mathbb{Q}$. Observe that special angles can certainly be trisected, e.g. just look at the trivial example $\theta = 0$.

In summary the bottom line is that angle trisection involves the solution of an irreducible polynomial of degree three whereas constructable numbers are only roots of polynomials of degree a power of two.

In the next section we show how angle trisection can be solved by Origami. This is in fact a folklore result in the community of mathematical paper folding and origami mathematics which has already been described in numerous papers. Nonetheless, we present the construction and prove its correctness again, we discuss the background from mathematical paper folding, we give further examples of classical problems that can be solved by Origami, and finally we also discuss some further methods of angle trisection using other tools. For a general account on geometric constructions we refer to [23] and two nice books on origami are [21, 14].

2. Angle trisection with Origami

Somewhat surprisingly and in contrast to constructions by compass and ruler angle trisection can be done by Origami (see e.g. [5, 8]). To start with we describe the rules that are allowed to fold a given sheet of paper, namely: All lines are defined by either the edge of the paper or a crease on the paper; all points are defined by the intersection of two lines; all folds must be uniquely defined by aligning combination of points and lines and a crease is formed by making a single fold, flattening the result and (optionally) unfolding. The last rule is somewhat restrictive, because it excludes multiple folds that are frequently used in more complicated Origami figures.

Now we give the construction of the angle trisection that was discovered by H. Abe (cf. [7]) and that was mentioned again at several places since then. We start with a square (or rectangular) paper and denote the corner points (starting with the top left and enumerating the others anti-clockwise) by A, B, C, D, respectively. Moreover, we assume that there is given a crease starting in B that meets the line segment AD in the point P (see Fig. 1).

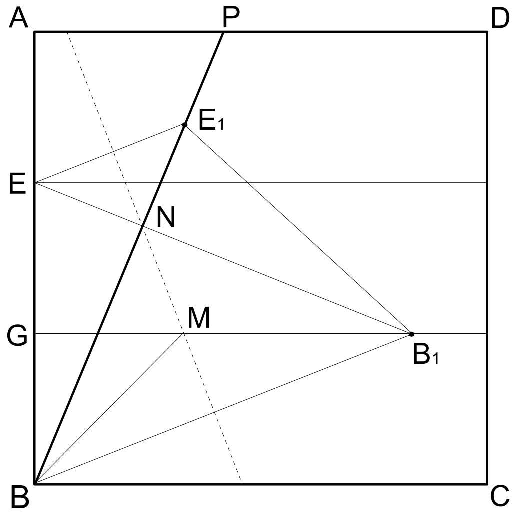

Fig. 1

Now we perform the following steps in order to trisect the angle $\theta = \sphericalangle(CBP)$:

Angle trisection with Origami

i) Make a crease by folding and unfolding edge BC parallel to AD; the point on AB is denoted by E.
ii) Fold edge BC up to the crease from step i) and unfold. The new point G is the mid-point of the segment BE.
iii) Make a fold such that the point E lies on the crease BP and simultaneously B lies on the crease obtained in step ii).
iv) Crease along the existing crease through point G, creasing through both layers.
v) Unfold.
vi) Extend the crease from step iv) to the point B and fold the edge BC to lie on this crease and unfold.

vii) The angle θ is trisected by $\sphericalangle(CBB_1)$.

Of course we still have to prove that this construction really works and we shall give two proofs of the following statement.

Theorem 1. *The Origami construction described above trisects the given angle θ.*

Geometric proof of the correctness. Let $\varphi = \sphericalangle(CBB_1)$. By construction the trapezoid BB_1E_1E is isosceles. Thus the triangle BB_1M is also isosceles. It follows that $\varphi = \sphericalangle(MB_1B) = \sphericalangle(B_1BM)$ where we have also used that the line starting at G is parallel to the line BC. Now we use the symmetry of the triangle EBB_1 to see that $\varphi = \sphericalangle(EB_1G)$. Finally, again by symmetry, the triangle BB_1N is also isosceles and therefore we get $\varphi = \sphericalangle(MBN)$. This shows that $\theta = \sphericalangle(CBP) = 3\varphi$ or $\varphi = \frac{\theta}{3}$. □

Alternatively we can introduce a coordinate system and then derive the same result by using polynomial equations that come from Abe's construction. We also give this proof.

Arithmetic proof of the correctness. Let B be the origin of the coordinate system and let BC lie on the x- and BA on the y-axis. Denote the coordinates of the yet unknown points E_1, B_1 by $E_1 = (\alpha, \beta), B_1 = (\gamma, \delta)$. Then we have $E = (0, 2\delta), B = (0, 0)$. We can still fix the unity of the coordinate system and we choose it such that $\gamma^2 + \delta^2 = 1$. This can be assumed without loss of generality. By construction it follows that $\alpha^2 + \beta^2 = 1$ and thus $B_1 = (\cos(\theta), \sin(\theta))$ where as before we have $\theta = \sphericalangle(CBP)$. We will use the following two conditions that are satisfied by construction: Firstly the line segment BE has the same length as B_1E_1, which gives the equation $4\delta^2 = (\alpha - \gamma)^2 + (\beta - \delta)^2$, and secondly EE_1 is parallel to BB_1, which gives

$$\frac{\alpha}{\beta - 2\delta} = \frac{\gamma}{\delta}.$$

Thus we have the following four relations:

$$\begin{aligned}
f_1 &= \gamma^2 + \delta^2 - 1 = 0\\
f_2 &= 3\delta^2 + 2\beta\delta - \alpha^2 + 2\alpha\gamma - \gamma^2 - \beta^2 = 0\\
f_3 &= \alpha\delta + 2\gamma\delta - \beta\gamma = 0\\
f_4 &= \alpha^2 + \beta^2 - 1 = 0
\end{aligned}$$

Our goal is to deduce an equation involving γ and α that is independent of β, δ. We get $3f_1 - f_2 = 0 = 4\gamma^2 - 2\alpha\gamma - 2\beta\delta + \alpha^2 + \beta^2 - 3$. By multiplying with $\alpha + 2\gamma$ and adding $2\beta f_3$ we get $8\gamma^3 - 6\gamma + \alpha^3 - 3\alpha + \alpha\beta^2 = 0$. Now we subtract αf_4 to get $8\gamma^3 - 6\gamma - 2\alpha = 0$. Dividing by 2 gives $4\gamma^3 - 3\gamma - \alpha = 0$. We have shown that γ is a root of the polynomial $4X^3 - 3X - \alpha$ with $\alpha = \cos(\theta)$ and thus we have indeed $\gamma = \cos(\frac{\theta}{3})$ as wanted. □

The strategy of variable elimination in the arithmetic proof is motivated by calculating the Gröbner basis of the ideal $\mathcal{I}$ generated by $f_1 = Z^2+W^2-1$, $f_2 = 3W^2+2YW-X^2+2XZ-Z^2-Y^2$, $f_3 = XW+2ZW-YZ$, $f_4 = X^2+Y^2-1$ with respect to some term order. From the proof we get

$$(3X+6Z)f_1-(X+2Z)f_2+2Yf_3-Xf_4 = 2(4Z^3-3Z-X).$$

We briefly give some more details: By using the lexicographic term order $\succ$ with $Y \succ W \succ X \succ Z$ we get $\{Y-4WZ^2+W, W^2+Z^2-1, X-4Z^3+3Z\}$ as the reduced Gröbner basis for $\mathcal{I}$; in particular this implies that f_1, f_2, f_3, f_4 do not generate the polynomial ring $\mathbb{Q}[X,Y,Z,W]$. We could have proved also other relations, e.g., taking the lexicographic term order with $X \succ Z \succ Y \succ W$ gives the reduced Gröbner basis $\{X+4ZW^2-Z, Z^2+W^2-1, Y+4W^3-3W\}$ for $\mathcal{I}$ and therefore we get $4\delta^3-3\delta+\beta=0$ which proves the correctness once again. The use of Gröbner basis is the algorithmic link between algebraization and proving that certain algebraic relations hold. We mention that such techniques are also used in [2].

Observe that there are also other Origami constructions that trisect a given angle (cf. [20, p. 34ff] and the footnote in [5, p. 285]). In conclusion we see that Origami can solve at least some equations of degree three. When looking at the steps in the construction of the Origami angle trisection, it is clear that the critical step is step iii). The dashed line in Fig. 1 is the crease needed for that step. With the notation of the arithmetic proof, the equation for this crease is given by $2(\gamma X+\delta Y)=1$ where γ satisfies the equation $4\gamma^3-3\gamma-\alpha=0$ and similarly δ satisfies the equation $4\delta^3-3\delta+\beta=0$. In other words finding that fold is equivalent to solve these particular cubic equations. The question arises which numbers can be constructed by using Origami. We shall discuss this in the next section.

3. Intermezzo on mathematical paper folding

The discussion on the constructability by Origami can be started similarly to the classification of constructable numbers in Section 1. We have already said what we mean by a line and a point and which folding operations are allowed. Let $\mathcal{N} \subseteq \mathbb{A}^2(\mathbb{R})$ be a set with $0, 1 \in \mathcal{N}$. We denote by $\overline{\mathcal{N}}$ the set of all points in the plane that can be constructed in a finite number of steps by the following operations:

(1) For any two points from $\overline{\mathcal{N}}$, there is a unique fold through both of them.
(2) Given two points $P_1, P_2 \in \overline{\mathcal{N}}$, then we can fold P_1 onto P_2.
(3) Given two lines defined over $\overline{\mathcal{N}}$, then we can fold one line to lie on the other one.
(4) Given a point $P \in \overline{\mathcal{N}}$ and a line defined over $\overline{\mathcal{N}}$, there is a fold perpendicular to the line passing through the point P.
(5) Given two points $P_1, P_2 \in \overline{\mathcal{N}}$ and a line defined over $\overline{\mathcal{N}}$, then there is a fold that places P_1 on the line such that the crease passes through P_2.

(6) Given two points $P_1, P_2 \in \overline{\mathcal{N}}$ and two lines defined over $\overline{\mathcal{N}}$, then there is a fold that places P_1 on the first and P_2 on the second line.
(7) Given a point $P \in \overline{\mathcal{N}}$ and two lines defined over $\overline{\mathcal{N}}$, then there is a fold perpendicular to the second line that places P on the first line.

The first six axioms were found by H. Huzita (cf. [18] and also [13]) and the seventh by K. Hatori and independently by J. Justin and by R. Lang. They are called the Huzita-Hatori axioms for origametric geometry (sometimes the Huzita-Justin axioms). It was proved by Lang that these axioms completely describe all operations that can be performed by paper folding (see [20]) and therefore they are now the fundamentals of the mathematical theory of paper folding.

We say that $z \in \mathbb{R}$ is Origami-constructable if the point $(z, 0)$ is in $\overline{\mathcal{N}}$. Let $\mathcal{O}(\mathcal{N})$ be the set of Origami-constructable numbers and $\mathbb{Q}(\mathcal{N})$ the field generated over $\mathbb{Q}$ by the coordinates of the points in $\mathcal{N}$. By examining the axioms it can be shown that the following theorem holds (we do not give all details in the proof):

Theorem 2. *The following statements are equivalent:*
(i) $z \in \mathcal{O}(\mathcal{N})$
(ii) *There is a sequence of quadratic or cubic field extensions* $K_0 := \mathbb{Q}(\mathcal{N}) \subset K_1 \subset K_2 \subset \cdots \subset K_n \subset \mathbb{R}$ *with* $z \in K_n$.
(iii) z *is contained in a Galois-extension* K *of* $\mathbb{Q}(\mathcal{N})$ *with* $[K : \mathbb{Q}(\mathcal{N})] = 2^a 3^b$ *for some non-negative integers* a, b.

Proof. The equivalence of (i) and (ii) follows from algebraization of the axioms, which shows that axioms (1)-(7) are equivalent to Origami-construct the roots of any irreducible polynomial of degree at most three with coefficients that are already Origami-constructable. For this see [20] or [1, 19].

The equivalence of (ii) and (iii) follows as in the classical case (cf. [3]): Given (ii) we just take the normal closure of K_n, which is a Galois-extension and of order divisible only by 2 and 3 since only elements of this order are used to generate it. Conversely, given (iii) we conclude by a famous theorem of Burnside (cf. [4]) that the Galois group of K is solvable. This means that there is a subnormal series whose quotients are cyclic groups of order 2 or 3. By Galois theory this precisely corresponds to a tower of field extensions as claimed in (ii). $\square$

We give a brief discussion on the axioms and their role concerning the quadratic and cubic field extensions above. The axioms (1)-(5) and (7) are equivalent to solve any quadratic equations with coefficients in $\mathcal{O}(\mathcal{N})$. In fact already the points that can be constructed by (1)-(5) are the same as the constructable points $\mathcal{K}(\mathcal{N})$ by compass and ruler. Moreover, (2),(3) and (5) are equivalent to (1)-(5). Axiom (7) just allows to solve certain quadratic equations. Axiom (6) however is equivalent to solve arbitrary cubic equations. In [20] this is deduced directly by studying the relevant algebraic equations. R. Alperin in [1] identified the pair given by a point and a line by a conic whose focus is the given point and directrix is the given line and uses some basic algebraic geometry. In this language (6) is equivalent to construct the simultaneous tangent line to the two parabolas with the given data of foci and directrices. From there it is also easy to see that conversely any cubic equation can by solved by this Origami step: Take the conics

$$\Big(Y - \frac{a}{2}\Big)^2 = 2bX, \quad Y = \frac{1}{2}X^2,$$

with Origami-constructable foci and directrices; then it is easy to verify that the common tangent has slope z satisfying $z^3 + az + b = 0$; hence we can solve any cubic equation with specified $a, b \in \mathcal{O}(\mathcal{N})$ (see [1, p. 129]).

We mention that Hatori has shown that in fact axiom (6) is enough and the others follow from it when we interpret a line placing P onto a given line on which P is already on either as a line perpendicular to it or a line passing through P. Finally, we remark that Alperin and Lang recently considered constructions where simultaneous folds are allowed and they formalized this in multi-fold axioms (cf. [2]); this in turn makes it possible to solve also algebraic equations of higher degree and to perform, for example, the quintisection of an angle.

In the next section we will discuss some further Origami constructions to well-known problems.

4. Further Origami constructions

Motivated by the fact that Origami can solve old problems we can look for other such results. Of course the antique problems of squaring or rectifying the circle are still out of reach since that would mean to construct the transcendental number π (but compare with [15]).

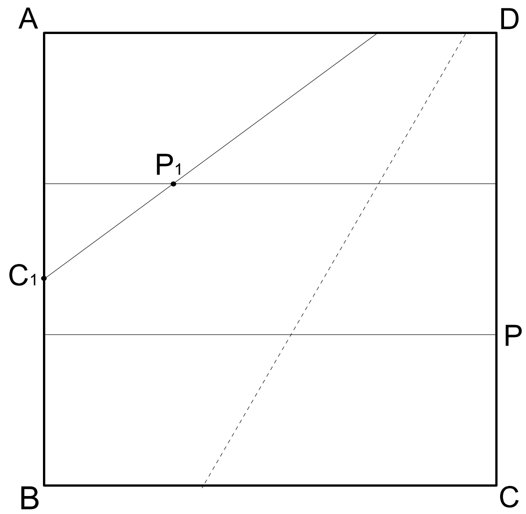

Fig. 2

However, the Delian problem of doubling the cube can be solved by Origami since it just means to construct the cube root of 2, i.e. the real solution to the cubic equation $X^3 - 2 = 0$, which in contrast is again not possible by compass and ruler. The construction is like this (see Fig. 2):

Squaring the cube with Origami

i) Take a square sheet of paper and divide it parallel to BC into three equal parts. We get two lines; the vertex of the lower line on the edge CD is denoted by P.
ii) Make a fold such that C is on the edge AB and the point P is on the upper line that divides BC into three parts.
iii) The ratio of lengths of AC_1 to C_1B is $\sqrt[3]{2}$.

This construction can be found in [24], where the correctness is also discussed. There are also other Origami constructions for $\sqrt[3]{2}$, e.g. based on solving the cubic equation $X^3 - 2 = 0$ (see [20]).

Another classical question is which regular polygons can be constructed by compass and ruler. It is well known (see [25]) that the regular n-gon is constructable if and only if $n = 2^k p_1 \cdots p_r$ for a non-negative integer k and distinct primes (called Fermat primes) of the form $p_j = 2^{k_j} + 1$ for $j = 1, \ldots, r$. Analogously, one can show (cf. [11, 6]) that a regular n-gon is Origami-constructable if and only if n is of the form $2^a 3^b p_1 \cdots p_s$ for non-negative integers a, b and distinct primes (called Pierpont primes) of the form $p_j = 2^{a_j} 3^{b_j} + 1$ for $j = 1, \ldots, s$. Equivalently, the regular n-gon is constructable if and only if $\varphi(n) = 2^l$ for a non-negative integer l, and Origami constructable if and only if $\varphi(n) = 2^c 3^d$ for non-negative integers c, d, where φ denotes Euler's totient function. An explicit folding construction can be found in [9] for $n = 7$ and in [10] for $n = 9$. We mention that this implies that Origami solves also certain higher degree equations like $X^7 - 1 = 0$ or $X^9 - 1 = 0$. Since we have Theorem 2 this of course implies that these equations can be reduced to solve cubic and quadratic equations only, for $X^7 - 1 = 0$ we see that $2\cos(\frac{2\pi}{7})$ is a root of $X^3 + X^2 - 2X - 1 = 0$ and similarly $\cos(\frac{2\pi}{9})$ is a root of $8X^3 - 6X + 1 = 0$. For more details we also refer to [5].

5. Angle trisection with other tools

Finally, we come back to our main topic and discuss some other methods to trisect a given angle. In fact it is well known (cf. [23]) that angles can be trisected if one slightly changes the rules of having compass and ruler. It goes back to Archimedes who used neusis constructions to trisect an angle by compass and a marked ruler. This construction is somewhat similar to what happens with Origami since there we have also marked a line segment (namely BE in the notation of Section 2) and moved it such that it appeared with certain properties (namely the vertices lie on certain lines).

The neusis construction of Archimedes on the other hand uses an anchor where the ruler is fixed and then it is arranged such that the marked segment is at a specific position.

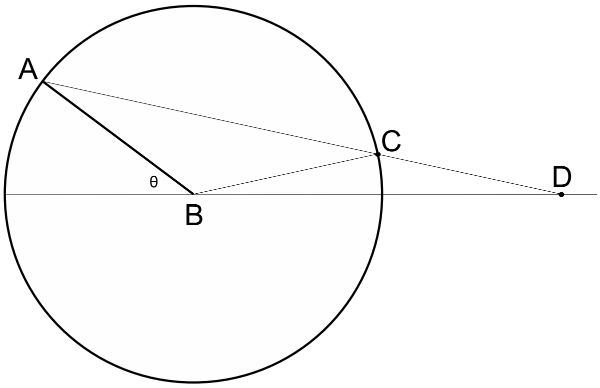

Fig. 3

Assume that the angle θ is given by the line AB and the horizontal line (see Fig. 3). Then the construction goes like follows:

Angle trisection with compass and marked ruler

i) Draw a circle with mid-point B and radius equal to the length of AB.
ii) Mark length AB at the ruler (if the ruler has a given mark then extend the line segment AB such that it has this length).
iii) Anchor the ruler at point A and move it until one end of the mark is on the circle and the other one is on the horizontal line.
iv) $\sphericalangle(ADB) = \frac{\theta}{3}$

It is not too hard to check that the construction really works. Thus by marking the ruler one can solve at least some cubic equations (namely those with all roots real). For more details see [12, Chapter 6, pp. 259ff]. In fact there it is shown that the set of real numbers that are constructable by using compass and a marked ruler is the same as the set of Origami-constructable numbers (cf. [11] and [12, Theorem 31.5 and Proposition 31.7]) since we have the following equivalence:

(i) $z \in \mathbb{R}$ is constructable in a finite number of steps by compass and marked ruler starting from a given set $\mathcal{M} \subseteq \mathbb{R}^2$ with $0, 1 \in \mathcal{M}$.
(ii) There is a sequence of quadratic or cubic field extensions $K_0 = \mathbb{Q}(\mathcal{M}) \subset K_1 \subset K_2 \subset \cdots \subset K_n \subset \mathbb{R}$ with $z \in K_n$.
(iii) There is a sequence of field extensions $K_0 := \mathbb{Q}(\mathcal{M}) \subset K_1 \subset K_2 \subset \cdots \subset K_n \subset \mathbb{R}$ where K_{i+1} is obtained from K_i by adjoining $\sqrt{a}$ with $a \in K_i, a > 0$, $\sqrt[3]{a}$ with $a \in K_i$ or $\cos(\frac{\theta}{3})$ with $\cos(\theta) \in K_i$ such that $z \in K_n$.
(iv) There is a Galois-extension K over $\mathbb{Q}(\mathcal{M})$ with $z \in K$ and $[K : \mathbb{Q}(\mathcal{M})] = 2^a 3^b$ for non-negative integers a, b.

This clearly also leads to the same conclusion for the construction of regular polygons as in the Origami case.

Other methods of trisecting an angle use curves other than circles; such curves are called trisectrices. For example one can use the limacon that is given in polar form by $r = \frac{1}{2} + \cos(\theta)$, the cycloid of Ceva that is given by $r = 1 + 2\cos(2\theta)$ or the quadratrix of Hippias that is given in implicit form by $x = y\cot(\frac{\pi}{2}y)$.

We briefly describe the construction by using the limacon (see [22, Part VI, 1.]): At the origin the curve has a double point and it consists of two closed loops the smaller one contained in the larger one. We put the angle θ that we want to trisect between the lines where the segment AB is on the x-axis and B is the origin.

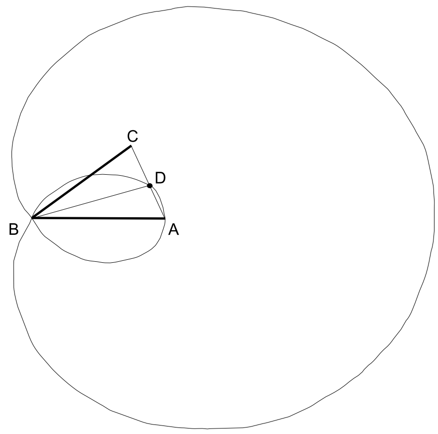

Fig. 4

Angle trisection with limacon

i) Fit AB such that it fits exactly in the smaller loop, i.e. the point A is on the limacon.
ii) Extend the line segment BC such that its length is equal to AB.
iii) Draw the line AC.
iv) The point D in which it intersects the inner loop of the limacon has the property $\theta = \sphericalangle(ABC) = 3\sphericalangle(ABD)$ and we have trisected the angle.

Even other tools like a protractor, a trisection tool, a tomahawk or a carpenter's square can be used and are in fact used in daily life for doing the job. For this and other methods we refer to [22] that gives a nice and amusing collection of trisection methods.

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